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Infinite-dimensional Representations of the Quantized Algebra $U(\mathfrak{gl}_N)$

Master's thesis

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1. Introduction

The main aim of the thesis is to construct some irreducible infinite-dimensional modules over the quantized algebra $U_q(\mathfrak{gl}_{n+m})$, which are unitarizable with respect to the real form $\mathfrak{u}(m,n)$.

Some special series of representations of the classical group of pseudo-unitary matrices $U(m,n)$ were described in the paper [5]. These representations were constructed using intuition from the representation theory of infinite-dimensiolnal groups. A basis of the vector space of such representations is given in terms of the Gelfand-Tsetlin formalism. The action of the Lie algebra $\mathfrak{gl}_{n+m}$ is given by the standard formulas for finite-dimensional irreducible representations. These infinite-dimensional representations admite q - analogs, which are described in the paper.

Representations from the present paper can be considered just as $U_q(\mathfrak{gl}_{n+m})$ modules. In this sense they are some special case of construction described in [1] and have some remarkable properties which we will use later. Note that, authors did not study unitarazibility of such representations.

The case $n = 1$ was considered in the paper [3]. We provide a construction that generalizes some results from [3] to the case $n \geq 2$.

Throughout the paper we assume $m \geq n$.

2. Finite-dimensional irreducible representations of $U_q(\mathfrak{gl}_N)$

Suppose q is a real-valued parameter, $0 < q < 1$.

For $x \in \mathbb{C}$ we denote by $[x]$ and $[x]_+$ the following q - numbers

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [x]_+ = \frac{q^x + q^{-x}}{q - q^{-1}}$$

Note that, $[x]_+ > 0$ for real x and $[x] = 0 \iff x \in \frac{\pi i}{\ln q} \cdot \mathbb{Z}$, where i is the imaginary unit.

For all $n \in \mathbb{N}$ we set $[n]! = [1][2], \ldots, [n]$ and $[0]! = 1$.

Definition 2.1. Algebra $U_q(\mathfrak{gl}_N)$ is an associative algebra with unit over the field of complex numbers with generators $K_1, \ldots, K_N, K_1^{-1}, \ldots, K_N^{-1}$, $E_1, \ldots, E_{N-1}, F_1, \ldots, F_{N-1}$ and following relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i E_j K_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} E_j, \; K_i F_j K_i^{-1} = q^{-\delta_{i,j} + \delta_{i,j+1}} F_j$$

$$[E_i, F_r] = \delta_{i,r} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}, \; [E_i, E_j] = [F_i, F_j] = 0, \text{ for } |i - j| \geq 2$$

$$E_i^2 E_{i\pm1} - [2] E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0, \; F_i^2 F_{i\pm1} - [2] F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

Let us fix a set of integers $\lambda = (\lambda_1, \ldots, \lambda_N)$ with the following condition $\lambda_i - \lambda_{i+1} \geq 0$ for $i = 1, 2, \ldots, N-1$.

Suppose that λ and μ are such sets of integers. We will call them *interlacing* and write $\mu \prec \lambda$, if the following condition is satisfied

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N$$

Consider the irreducible finite-dimensional $U_q(\mathfrak{gl}_N)$ - module with the highest weight $q^\lambda \overset{def}{=} (q^{\lambda_1}, \ldots, q^{\lambda_N})$. Denote this module by $L(\lambda)$.

The restriction of $L(\lambda)$ to the subalgebra of $U_q(\mathfrak{gl}_N)$ isomorphic to $U_q(\mathfrak{gl}_{N-1})$ and generated by $K_1, \ldots, K_{N-1}, K_1^{-1}, \ldots, K_{N-1}^{-1}$, $E_1, \ldots, E_{N-2}, F_1, \ldots, F_{N-2}$ satisfies the following

Proposition 2.2 ([7], Proposition 2.12)**.**

$$\operatorname{Res}^N_{N-1} L(\lambda) = \bigoplus_{\mu \prec \lambda} L(\mu)$$

Since all irreducible representations of $U_q(\mathfrak{gl}_1)$ are one-dimensional we obtain a basis in $L(\lambda)$ consisting of Gelfand-Tsetlin tableaux with fixed top row that is equal to λ [4].

We will denote by Λ these tableaux. The element from Λ with coordinates (i, k) will be denoted by $\lambda_{i,k}$. Here i is the colomn index and k is the row index. Rows and colomns are numbered starting from the bottom and left respectively. We denote the basis vector corresponding to Gelfand-Tsetlin tableau Λ by ξ_Λ.

Theorem 2.3 ([4], 7.3.3, Theorem 24)**.** *The action of generators of the quantiazed algebra $U_q(\mathfrak{gl}_N)$ in the irreducible finite-dimensional representation $L(\lambda)$ is given by the following formulas in the basis of Gelfand-Tsetlin tableaux.*

$$K_k \xi_\Lambda = q^{a_k(\Lambda)} \xi_\Lambda, \; \textit{where} \;\; a_k(\Lambda) = \sum_{i=1}^{k} \lambda_{i,k} - \sum_{i=1}^{k-1} \lambda_{i,k-1}, \; \textit{for } 1 \leq k \leq N \tag{2.1}$$

$$E_k \xi_\Lambda = \sum_{j=1}^{k} a_{j,k}(\Lambda) \xi_{\Lambda + \varepsilon_j(k)}, \quad F_k \xi_\Lambda = \sum_{j=1}^{k} b_{j,k}(\Lambda) \xi_{\Lambda - \varepsilon_j(k)}, \quad \textit{for } 1 \leq k \leq N-1$$

$$a_{j,k}(\Lambda) = -\frac{\prod_{i=1}^{k+1}[l_{i,k+1} - l_{j,k}]}{\prod_{i=1, i\neq j}^{k}[l_{i,k} - l_{j,k}]}, \quad b_{j,k}(\Lambda) = \frac{\prod_{i=1}^{k-1}[l_{i,k-1} - l_{j,k}]}{\prod_{i=1, i\neq j}^{k}[l_{i,k} - l_{j,k}]}, \tag{2.2}$$

where $l_{i,j} = \lambda_{i,j} - i$

We use the convention that if $\Lambda \pm \varepsilon_j(k)$ *is not a Gelfand-Tsetlin tableau, then the corresponding coefficient of this summand is equal to zero.*

3. Infinite-dimensional representations of $U_q(\mathfrak{gl}_N)$

Let us fix some positive integers n, m, where $m \geq n \geq 1$ and let $N = n + m$.

Consider the set of complex numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of a special type

$$\lambda = (z_1 + \lambda'_1, z_1 + \lambda'_2, \ldots, z_1 + \lambda'_n; z_3 + \lambda'_{n+1}, \ldots, z_3 + \lambda'_m; z_2 + \lambda'_{m+1}, z_2 + \lambda'_{m+2}, \ldots, z_2 + \lambda'_N),$$

where $z_1, z_2, z_3 \in \mathbb{C}$ and the following condition is satisfied

$$z_1 - z_3, z_2 - z_3, z_1 - z_2 \notin \mathbb{Z} + \frac{\pi i}{\ln q} \cdot \mathbb{Z} \tag{3.1}$$

Numbers λ'_i are assumed to be integers with the following non-increasing condition

$$\lambda'_i - \lambda'_{i+1} \in \mathbb{Z}_{\geq 0} \text{ for all } i \neq n, m \tag{3.2}$$

In the case $m = n$ we treat z_3 as an extra parameter, the sense of which will become clear after Definition 3.1.

We will call such an N - tuple λ *an admissible top row corresponding to the pair* (n, m).

Suppose we have two such N - tuples λ and μ that are admissible top rows corresponding to pairs (n, m) and $(n-1, m)$ respectively.

We will say that λ and μ *interlace* and write $\mu \prec \lambda$ if their parameters z_i coincide and the following condition is fulfilled

$$\lambda'_1 \geq \mu'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_{n-1} \geq \mu'_{n-1} \geq \lambda'_n$$
$$\mu'_n \geq \lambda'_{n+1} \geq \mu'_{n+1} \geq \ldots \geq \mu'_{m-1} \geq \lambda'_m \geq \mu'_m$$
$$\lambda'_{m+1} \geq \mu'_{m+1} \geq \lambda'_{m+2} \geq \ldots \geq \lambda'_{N-1} \geq \mu'_{N-1} \geq \lambda'_N$$

Further we will consider triangular tableaux of Gelfand-Tsetlin type, which are associated to an admissible top row λ. Rows and coloumns in such tableaux are indexed as in the finite-dimensional case.

Definition 3.1. Suppose that λ is an admissible top row corresponding to the pair (n, m). We will say that the triangular tableau with complex numbers and the top row λ is a modified Gelfand-Tsetlin tableau iff every two rows in such tableau are interlacing.

In the case $n = m$ we suppose that all numbers in the third part of the tableau are equal modulo 1 to each other. We will denote this residue by z_3.

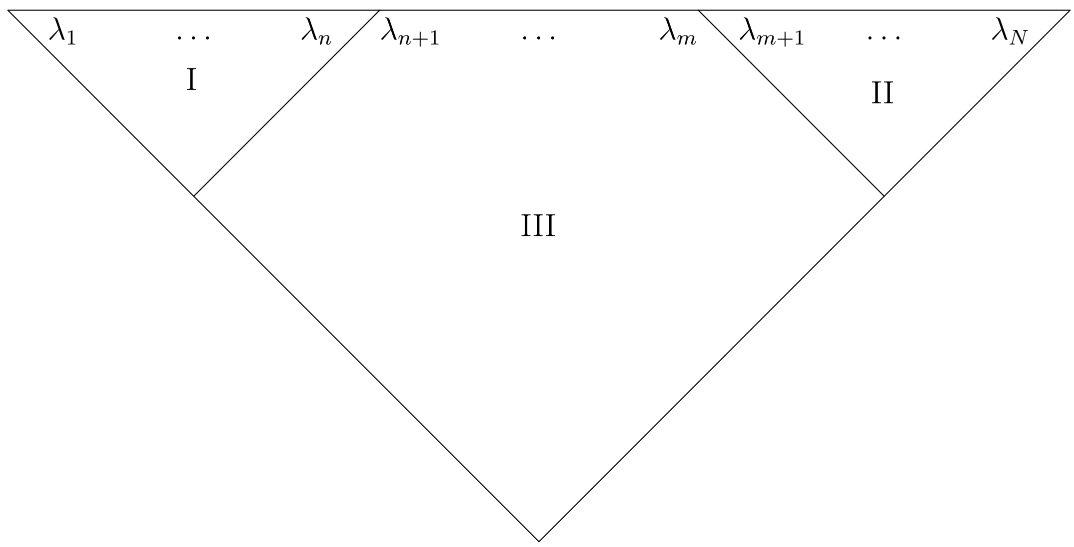

Figure 1. Modified Gelfand-Tsetlin tableau

Note that, inside the first, second and the third parts of the tableau standard Gelfand-Tsetlin inequalities will hold for the real part of tableau's elements, but there is no links between different parts of the tableau (see Figure 1).

Denote by S_λ the set of all modified Gelfand-Tsetlin tableaux with the fixed admissible top row λ.

Consider the countable-dimensional vector space $V_\lambda = \mathrm{span}\,(\xi_\Lambda \mid \Lambda \in S_\lambda)$.

In the space V_λ, we define the action of the algebra $U_q(\mathfrak{gl}_N)$ on the basis of modified G-T tableaux by formulas (2.1), (2.2). It is assumed that coefficients $a_{j,k}$ and $b_{j,k}$ in formulas (2.1), (2.2) are zero by definition if $\Lambda \pm \varepsilon_j(k)$ is not an admissible tableau.

The denominators of coefficients in formulas (2.2) do not vanish due to the condition (3.1).

Theorem 3.2. *Suppose that λ is an admissible top row. Then formulas* (2.1), (2.2) *define a representation of $U_q(\mathfrak{gl}_N)$ on V_λ.*

Remark 3.3. Apparently, representations of such a type were firstly obtained in the work [5]. It was done using some intuition from representation theory of infinite-dimensional classical groups. Representations in the present paper can be considered as their natural q - analogs.

Note that, a much more general construction of $U_q(\mathfrak{gl}_N)$ modules was obtained in the paper [1], but authors did not study unitarazibility of such representations.

Proof. We will fix $\lambda' = (\lambda'_1, \ldots, \lambda'_N)$ and vary parameters z_1, z_2, z_3. Let R be a noncommutative polynomial in the generators, which vanishes in $U_q(\mathfrak{gl}_N)$. Apply R to some basis vector ξ_Λ, corresponding to the tableau $\Lambda \in S_\lambda$. We get a linear combination of basis vectors with coefficients that are rational functions in two variables $q^{z_1-z_3}$ и $q^{z_3-z_2}$. We will show that all such coefficients, in fact, vanish.

We introduce the following notation: $t_1 = z_1 - z_3$, $t_2 = z_3 - z_2$

$$\gamma^+ = \left\{(t_1, t_2) \in \mathbb{Z}^2 | t_1 \geq \lambda_{n+1} - \lambda_n,\ t_2 \geq \lambda_{m+1} - \lambda_m\right\}$$

Denote one of such coefficients by $C\left(q^{t_1}, q^{t_2}\right)$, where $C(x, y)$ is a rational function.

Note that, the top row of tableaux is the highest weight of a finite-dimensional representation for all $(t_1, t_2) \in \gamma^+$. Moreover, if values t_1, t_2 increase inside of γ^+, then such finite-dimensional representation is growing and contain more and more tableaux from V_λ.

Therefore, for sufficiently large values $(t_1, t_2) \in \gamma^+$, the formulas for the action of R on the ξ_Λ in such finite-dimensional representation are the same as (2.1), (2.2).

Denote the set of all such (t_1, t_2) by the symbol γ^+_Λ. Then for all $(t_1, t_2) \in \gamma^+_\Lambda$ we have $C\left(q^{t_1}, q^{t_2}\right) = 0$.

It is clear, that γ^+_Λ contains a two-dimensional $\mathbb{Z}_{\geq 0}$ - lattice. Therefore, $C\left(q^{t_1}, q^{t_2}\right)$ is identically zero. □

The representation afforded by Theorem 3.2 will be denoted by T_λ.

4. Unitarizable representations of $U_q(\mathfrak{u}(m,n))$

The main aim of this section is to introduce three series of unitarizable representations of $U_q(\mathfrak{u}(m,n))$ of the form T_λ.

There is an involution $*$ on the algebra $U_q(\mathfrak{gl}_N)$, which endows it with the structure of a $*$ - algebra.

This involution is defined on the generators as follows

$$K_i^* = K_i, \quad E_i^* = (-1)^{\delta_{m,i}} F_i, \quad F_i^* = (-1)^{\delta_{m,i}} E_i,$$

where $\delta_{m,i}$ is the Kronecker delta.

Definition 4.1. The algebra $U_q(\mathfrak{u}(m,n))$ is the $*$ - algebra $(U_q(\mathfrak{gl}_N), *)$, where $*$ is defined as above.

Definition 4.2. A representation T of the algebra $U_q(\mathfrak{gl}_N)$ on the space V is called $\mathfrak{u}(m,n)$ unitarizable, if it is equivalent to a $*$ - representation of $U_q(\mathfrak{u}(m,n))$. More precisely, there is an Hermitian scalar product in the space V, such that

$$(T(a)v_1, v_2) = (v_1, T(a^*)v_2), \quad \forall a \in U_q(\mathfrak{gl}_N), \ \forall v_1, v_2 \in V. \tag{4.1}$$

It is assumed that the Hermitian scalar product is linear with respect to the first argument and antilinear in the second one.

Now we focus only on unitarizable representations of $U_q(\mathfrak{u}(m,n))$ of the form T_λ.

Consider a representation T_λ and define a scalar product on V_λ in the following way

$$(\xi_\Lambda, \xi_\Lambda) = H(\Lambda), \ \forall \Lambda \in S_\lambda,$$

where $H(\cdot)$ is a strictly positive function on the set of modified Gelfand-Tsetlin tableaux with the top row is equal to λ.

Suppose that $(\cdot,\cdot)$ is $U_q(\mathfrak{u}(m,n))$ - invariant in the sense of the definition above. Then we immediately conclude that

$$\frac{H(\Lambda + \varepsilon_j(k))}{H(\Lambda)} = (-1)^{\delta_{m,k}} \frac{\overline{b_{j,k}(\Lambda + \varepsilon_j(k))}}{a_{j,k}(\Lambda)} = (-1)^{\delta_{m,k}} \frac{b_{j,k}(\Lambda + \varepsilon_j(k))}{\overline{a_{j,k}(\Lambda)}}, \tag{4.2}$$

for all $\Lambda \in S_\lambda$ such that $\Lambda + \varepsilon_j(k) \in S_\lambda$.

Note that, the strict positivity of the right hand side of the equality (4.2) and the existence of a solution of this reccurence relations are sufficient conditions for unitarazibility of the representation T_λ.

We need to introduce some notation. Define the following expression for all modified G-T tableaux Λ.

$$N(\Lambda) = \prod_{k=2}^{N} \left(\prod_{1\le i<j\le k} \frac{\Gamma_q(l_{i,k} - l_{j,k})}{\Gamma_q(l_{i,k-1} - l_{j,k})} \prod_{1\le i\le j\le k-1} \frac{\Gamma_q(l_{i,k} - l_{j,k-1} + 1)}{\Gamma_q(l_{i,k-1} - l_{j,k-1} + 1)} \right), \tag{4.3}$$

where Γ_q is the q - gamma function which is defined as follows [4]:

$$\Gamma_q(z) = q^{-\frac{(z-1)(z-2)}{2}} (1-q^2)^{1-z} \frac{(q^2;q^2)_\infty}{(q^{2z};q^2)_\infty},$$

where $(a;q)_\infty = \prod_{j=1}^{\infty} (1 - aq^{j-1})$ is the Pochhammer's symbol.

The q - gamma function satisfies the functional equation

$$\Gamma_q(z+1) = [z]\Gamma_q(z)$$

Remark 4.3. The expression (4.3) satisfies the following identity

$$\frac{N(\Lambda + \varepsilon_j(k))}{N(\Lambda)} = \frac{b_{j,k}(\Lambda + \varepsilon_j(k))}{a_{j,k}(\Lambda)}. \tag{4.4}$$

Now we are ready to give an explicit construction of three series of unitarizable $U_q(\mathfrak{u}(m,n))$ modules.

Suppose that we have a set of real numbers $\lambda \in \mathbb{R}^N$ that satisfy the following conditions

$$\begin{aligned} &\bullet\ \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \text{ for all } i \neq n, m \\ &\bullet\ \lambda_1 - \lambda_N \notin \mathbb{Z} \end{aligned} \tag{4.5}$$

We set

$$\lambda^0 = (\lambda_1 + it_0, \lambda_2 + it_0, \ldots, \lambda_n + it_0; \lambda_{n+1}, \ldots, \lambda_m; \lambda_{m+1} - it_0, \lambda_{m+2} - it_0, \ldots, \lambda_N - it_0),$$

where $t_0 = \frac{\pi i}{\ln q}$.

Then λ^0 is an admissible top row corresponding to the pair (n, m). We denote the representation T_{λ^0} by T_λ^s.

Proposition 4.4. *If $n = 1$, then the representation T_λ^s of the agebra $U_q(\mathfrak{u}(m,n))$ is unitarizable.*

For $n \geq 2$ we have the following sufficient condition for T_λ^s to be unitarizable

$$m > \lambda_{m+1} - \lambda_n + n - 2 \tag{4.6}$$

Remark 4.5. In the case $n \geq 2$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, $\lambda_{m+1} = \lambda_{m+2} = \ldots = \lambda_N$ the sufficient condition above is not necessary and the representation T_λ^s will be unitarizable for all admissible values of parametres λ that satisfy (4.5).

Remark 4.6. Representations T_λ^s disappear in the classical limit $q \to 1$.

Proof. In this proof we denote elements of $\Lambda = \mathrm{Re}\,(\Lambda^0)$ by $\lambda_{j,k}$ for $\Lambda^0 \in S_{\lambda^0}$.

Coefficients in formulas 2.2 for such representations we denote by $a^s_{j,k}(\Lambda)$ and $b^s_{j,k}(\Lambda)$.

Note that, $a^s_{j,k}(\Lambda) = a_{j,k}(\Lambda)$ for $k \leq m-1$ and $b^s_{j,k}(\Lambda) = b_{j,k}(\Lambda)$ for $k \leq m$.

For all $x \in \mathbb{R}$ the following equalities hold

$$[x \pm ti] = \pm i[x]_+, \quad [x \pm 2ti] = -[x] \tag{4.7}$$

Now it's clear, that $b^s_{j,k}(\Lambda) \in \mathbb{R}$ if the pair of indices (j,k) lies in the third part of the tableau and $b^s_{j,k}(\Lambda) \in i \cdot \mathbb{R}$ if the pair (j,k) lies in the first or in the second part of the tableau.

Then we set

$$H(\Lambda) = \prod_{(i,j) \in I \cup II} (-1)^{\lambda_{i,j}} \cdot (-1)^{\sum_{i=1}^m \lambda_{i,m}} \cdot N(\Lambda^0)$$

We need explicit expressions of coefficients $a^s_{j,k}(\Lambda)$ and $b^s_{j,k}(\Lambda)$. They are as follows

$$a^s_{j,m}(\Lambda) = -[l_{1,m+1} - l_{j,m}]_+ [l_{m+1,m+1} - l_{j,m}]_+ \frac{\prod_{i=2}^{m} [l_{i,m+1} - l_{j,m}]}{\prod_{i=1, i\neq j}^{m} [l_{i,m} - l_{j,m}]} \tag{4.8}$$

$$a^s_{j,k}(\Lambda) = -\frac{\prod_{i:(i,k+1)\in I\cup II} [l_{i,k+1} - l_{j,k}]_+}{\prod_{i:(i,k)\in I\cup II} [l_{i,k} - l_{j,k}]_+} \frac{\prod_{i:(i,k+1)\in III} [l_{i,k+1} - l_{j,k}]}{\prod_{\substack{i:(i,k)\in III \\ i\neq j}} [l_{i,k} - l_{j,k}]}, \tag{4.9}$$

if $(j,k) \in III$.

$$a^s_{j,k}(\Lambda) = \pm i \cdot \frac{\prod_{i:(i,k+1)\in III} [l_{i,k+1} - l_{j,k}]_+}{\prod_{i:(i,k)\in III} [l_{i,k} - l_{j,k}]_+} \frac{\prod_{i:(i,k+1)\in I\cup II} [l_{i,k+1} - l_{j,k}]}{\prod_{\substack{i:(i,k)\in I\cup II \\ i\neq j}} [l_{i,k} - l_{j,k}]}, \tag{4.10}$$

if $(j,k) \in I \cup II$.

In the last expression the plus sign is taken if $(j,k) \in I$, and minus sign is taken if $(j,k) \in II$.

$$b^s_{j,m+1}(\Lambda) = \frac{1}{[l_{1,m+1} - l_{j,m+1}]_+ [l_{m+1,m+1} - l_{j,m+1}]_+} \cdot \frac{\prod_{i=1}^{m} [l_{i,m} - l_{j,m+1}]}{\prod_{\substack{i=2 \\ i \neq j}}^{m} [l_{i,m+1} - l_{j,m+1}]}, \tag{4.11}$$

if $(j, m+1) \in III$.

$$b^s_{1,m+1}(\Lambda) = i \cdot \frac{1}{[l_{m+1,m+1} - l_{1,m+1}]} \frac{\prod_{i=1}^{m} [l_{i,m} - l_{1,m+1}]_+}{\prod_{i=2}^{m} [l_{i,m+1} - l_{1,m+1}]_+} \tag{4.12}$$

$$b^s_{m+1,m+1}(\Lambda) = -i \cdot \frac{1}{[l_{1,m+1} - l_{m+1,m+1}]} \frac{\prod_{i=1}^{m} [l_{i,m} - l_{m+1,m+1}]_+}{\prod_{i=2}^{m} [l_{i,m+1} - l_{m+1,m+1}]_+} \tag{4.13}$$

$$b^s_{j,k}(\Lambda) = \frac{\prod_{i:(i,k-1) \in I \cup II} [l_{i,k-1} - l_{j,k}]_+}{\prod_{i:(i,k) \in I \cup II} [l_{i,k} - l_{j,k}]_+} \frac{\prod_{i:(i,k-1) \in III} [l_{i,k-1} - l_{j,k}]}{\prod_{\substack{i:(i,k) \in III \\ i \neq j}} [l_{i,k} - l_{j,k}]}, \tag{4.14}$$

if $(j,k) \in III$.

$$b^s_{j,k}(\Lambda) = \pm i \cdot \frac{\prod_{i:(i,k-1) \in III} [l_{i,k-1} - l_{j,k}]_+}{\prod_{i:(i,k) \in III} [l_{i,k} - l_{j,k}]_+} \frac{\prod_{i:(i,k-1) \in I \cup II} [l_{i,k-1} - l_{j,k}]}{\prod_{\substack{i:(i,k) \in I \cup II \\ i \neq j}} [l_{i,k} - l_{j,k}]}, \tag{4.15}$$

if $(j,k) \in I \cup II$.

The rule for signs is the same as for (4.10).

Now we show that the condition (4.6) is sufficient for the following inequality to be fulfilled

$$\forall \Lambda^0 \in S_{\lambda^0} \colon \Lambda^0 + \varepsilon_j(k) \in S_{\lambda^0} \quad (-1)^{\delta_{m,k}} \frac{\overline{b^s_{j,k}(\Lambda + \varepsilon_j(k))}}{a^s_{j,k}(\Lambda)} > 0 \tag{4.16}$$

Note that, this inequality holds for $k \leq m$ because of standard Gelfand-Tsetlin inequalities for elements of the third part of tableaux. Moreover, it is easy to prove, that this inequality is satisfied for $k \geq m+1$, $(j,k) \in III$.

It is clear, that if $n = 1$, then T^s_λ is unitarizable for all admissible values of parameter λ, and the statement from the remark 4.5 is fulfilled.

We see that the property of T^s_λ to be unitarizable does not depend on the third part of the tableaux at all.

From (4.16) we have the following conditions which we need to verify.

$$\frac{\prod_{i=m+1}^{k-1} (l_{i,k-1} - l_{j,k} - 1) \prod_{i=m+1}^{k+1} (l_{i,k+1} - l_{j,k})}{\prod_{i=m+1}^{k} (l_{i,k} - l_{j,k} - 1)(l_{i,k} - l_{j,k})} > 0, \tag{4.17}$$

for $k \geq m+2$ and $j = \overline{1, k-m}$.

$$\frac{\prod_{i=1}^{k-m-1}(l_{i,k-1}-l_{j,k}-1)\prod_{i=1}^{k-m+1}(l_{i,k+1}-l_{j,k})}{\prod_{i=1}^{k-m}(l_{i,k}-l_{j,k}-1)(l_{i,k}-l_{j,k})}>0, \tag{4.18}$$

for $k \geq m+2$ and $j=\overline{m+1,k}$.

$$\begin{gathered}(l_{m+1,m+1}-l_{1,m+1})(l_{m+1,m+1}-l_{1,m+1}-1)\cdot\\ (l_{m+1,m+2}-l_{1,m+1})(l_{m+2,m+2}-l_{1,m+1})>0,\end{gathered} \tag{4.19}$$

for $(j,k)=(1,m+1)$.

$$\begin{gathered}(l_{1,m+1}-l_{m+1,m+1})(l_{1,m+1}-l_{m+1,m+1}-1)\cdot\\ (l_{1,m+2}-l_{m+1,m+1})(l_{2,m+2}-l_{m+1,m+1})>0,\end{gathered} \tag{4.20}$$

for $(j,k)=(m+1,m+1)$.

Now we check that the condition $m > \lambda_{m+1}-\lambda_n+n-2$ is suffucient for (4.17),(4.18),(4.19),(4.20) to be satisfied for $n \geq 2$.

For (4.19) we need the following inequalities

$$l_{1,m+1} \geq \lambda_n - 1, \quad l_{m+1,m+1} \leq \lambda_{m+1}-m-1$$

We immediately get

$$l_{m+1,m+1}-l_{1,m+1} \leq \lambda_{m+1}-\lambda_n-m-m < 2-n \leq 0$$

We alse have, $l_{m+1,m+1}-l_{1,m+1}-1<0$. Then we proceed similarly with the remaining two factors in (4.19) and show that each of them is negative. We are done with (4.19).

For (4.20) we need the following

$$\begin{aligned}l_{1,m+1}-l_{m+1,m+1}-1 &\geq \lambda_n-1-(\lambda_{m+1}-1)+m+1-1=\\ &=\lambda_n-\lambda_{m+1}+m>n-2\geq 0,\end{aligned} \tag{4.21}$$

$$\begin{aligned}l_{2,m+2}-l_{m+1,m+1} &\geq \lambda_n-2-(\lambda_{m+1}-1)+m+1=\\ &=\lambda_n-\lambda_{m+1}+m>n-2\geq 0\end{aligned} \tag{4.22}$$

For (4.17) we use

$$\begin{gathered}l_{i,k}-l_{j,k} \leq \lambda_{m+1}-i-\lambda_n+j<m-n-2-i+j\leq\\ \leq m-n-2-(m+1)+k-m=k-m-n-3<0,\end{gathered} \tag{4.23}$$

$$l_{i,k+1}-l_{j,k} \leq \lambda_{m+1}-i-\lambda_n+j<0, \tag{4.24}$$

$$l_{i,k-1}-l_{j,k}-1 \leq \lambda_{m+1}-i-\lambda_n+j-1<0 \tag{4.25}$$

And for (4.18) we use

$$\begin{gathered}l_{i,k}-l_{j,k}-1 \geq \lambda_n-i-(\lambda_{m+1}-1)+j-1=\lambda_n-\lambda_{m+1}-i+j>\\ >n-2-m-i+j>k-m+n-1\geq m+2-m+n-1=n+1>0\end{gathered} \tag{4.26}$$

$$l_{i,k+1}-l_{j,k} \geq \lambda_n-i-(\lambda_{m+1}-1)+j>0 \tag{4.27}$$

$$l_{i,k-1}-l_{j,k}-1 \geq \lambda_n-i-(\lambda_{m+1}-1)+j-1>0 \tag{4.28}$$

Now we are done with it and we are going to show that such representations do not sustain passage to the limit as $q \to 1$.

To do that, we use the orthonormal basis $\xi'_\Lambda = \frac{\xi_\Lambda}{\sqrt{H(\Lambda)}}$ and we look at the coefficient

$$\left(E_m\xi'_\Lambda, \xi'_{\Lambda+\varepsilon_j(m)}\right) = -\sqrt{-a^s_{j,m}(\Lambda)b^s_{j,m}(\Lambda+\varepsilon_j(m))}$$

It is easy to see from (4.8) that the expression from the right hand side of the equality does not have the classical limit. $\square$

Consider the second family of unitarizable representations.

Let us fix two non-negative integers k, l such that $m \geq 2n+k+l$. Suppose that λ is an admissible top row corresponding to the pair (n,m), which is of the form

$$\lambda = (\lambda_1, \dots, \lambda_n; \lambda_{n+1}, \dots, \lambda_{n+k}, 0, \dots, 0, \lambda_{m-l+1}, \dots, \lambda_m; \lambda_{m+1}, \dots, \lambda_N) \in \mathbb{R}^N$$

The representation T_λ we denote by T^c_λ.

Proposition 4.7. *If the following conditions for λ are satisfied*

$$-\lambda_1 > n+k-1,\ \lambda_N > n+l-1, \tag{4.29}$$

$$m > \lambda_{m+1} - \lambda_n + n - 2, \tag{4.30}$$

then T^c_λ is a unitarizable representation.

Moreover, in the case $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, $\lambda_{m+1} = \lambda_{m+2} = \ldots = \lambda_N$ conditions (4.29), (4.29) *are not necessary. In this case we have a weaker sufficient conditions*

$$-\lambda_n < m-n-l+1, \quad \lambda_{m+1} < m-n-k+1 \tag{4.31}$$

Remark 4.8. If $n=1$, we can drop the condition $m \geq 2n+k+l$ and require the fulfillment of (4.29) and (4.31). Then T^c_λ is a unitarizable representation.

Remark 4.9. Representations from Proposition 4.7 can be considered as q - analogs of representations from the paper [5]. All sufficient conditions for quantized case are absolutely the same as for the classical one.

Now we describe the third series of unitarizable representations of $U_q(\mathfrak{u}(m,n))$.

Consider an admissible top row λ corresponding to the pair (n,m) which is of the following form

$$\lambda = (a+xi, \dots, a+xi; \lambda_n, \dots, \lambda_m; a+m-xi, \dots, a+m-xi),$$

where $\lambda_n, \dots, \lambda_m \in \mathbb{R}$, $a \in \mathbb{R}$, $x \in \left(0; -\frac{\pi}{\ln q}\right)$, $x \neq -\frac{\pi}{2\ln q}$.

The representation T_λ we denote by T^p_λ.

Proposition 4.10. *The representation T^p_λ is unitarizable.*

Proof. Note that, coefficients $a_{j,k}(\Lambda)$ and $b_{j,k}(\Lambda)$ for such representation are products of factors of the form

$$[y+ix][y-ix] = [y]^2 + \frac{4\sin^2(x\ln q)}{(q-q^{-1})^2} > 0$$

for some $y \in \mathbb{R}$.

It follows that $\overline{b_{j,k}(\Lambda)} = b_{j,k}(\Lambda)$. Then it is easy to see that the desired inequality for the right hand side of (4.2) holds. □

5. Some properties of representations T_λ

In this section we will show some properties of representations T_λ, including their irreducibility and we will see that T^c_λ, T^s_λ and T^p_λ are pairwise non-equivalent for different values of parameters.

Example 5.1. Consider the representation T_λ of the algebra $U_q(gl_2)$. The top row λ is of the following form $\lambda = (a + xi; a + 1 - xi)$, where $a \in \mathbb{R}\backslash\mathbb{Z}$, $x \notin \frac{\pi}{2\ln q}\mathbb{Z}$. Parameter z_3 is equal to zero (3.1).

The action of $U_q(gl_2)$ is determined by the formulas

$$K_1\xi_\Lambda = q^{\lambda_{1,1}},\ K_2\xi_\Lambda = q^{2a+1-\lambda_{1,1}} \tag{5.1}$$

$$E_1\xi_\Lambda = -[a - \lambda_{1,1} + xi][a - \lambda_{1,1} - xi]\xi_{\Lambda+\varepsilon_1(1)},\ F_1\xi_\Lambda = \xi_{\Lambda-\varepsilon_1(1)} \tag{5.2}$$

It is clear that these formulas are invariant under the transformation $x \to -x$. Moreover, the eigenvalues of elements from the Cartan subalgeba $\mathbb{C}\left[K_1^\pm, K_2^\pm\right]$ do not depend on x at all. But the quadratic Casimir

$$C = E_1F_1 + \frac{q^{-1}K_1K_2^{-1} + qK_1^{-1}K_2}{(q-q^{-1})^2} - 2 = F_1E_1 + \frac{qK_1K_2^{-1} + q^{-1}K_1^{-1}K_2}{(q-q^{-1})^2} - 2$$

acts in such representation as multiplication by a constant which is equal to $[xi]^2$. This implies that such representations could not be equivalent for all values of x.

The action of Cartan subalgebra does not separate such representations into equivalence classes, but some other commutative subalgebra of $U_q(\mathfrak{gl}_2)$ does that: $\Gamma^2_q = \langle K_1, K_2, C\rangle = \langle Z_1, Z_2\rangle$, where we denote the center of $U_q(\mathfrak{gl}_i)$ by Z_i . The situation is a little bit similar in the general case for representations T_λ.

Let Z_N denote the center of $U_q(\mathfrak{gl}_N)$. The algebra Z_N is finitely generated by some special elements $c_{N,0}, \dots, c_{N,N}$ [6, Theorem 14].

Consider the chain of algebras

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \dots \subset U_q(\mathfrak{gl}_N), \tag{5.3}$$

where the inclusions are the same as was pointed in section 2.

By the very construction, Γ^N_q is a commutative subalgebra of $U_q(\mathfrak{gl}_N)$ that is generated by $Z_1, \dots, Z_N$. It is called *the Gelfand-Tsetlin subalgebra.*

Proposition 5.2. [2, Theorem 4.3] *The action of Γ^N_q in the irreducible finite-dimensional representation $L(\lambda)$ is diagonizable in the basis of Gelfand-Tsetlin tableaux:*

$$c_{r,s}(\xi_\Lambda) = \gamma_{r,s}(\Lambda)\xi_\Lambda,\ \textit{for } r \le N,\ 0 \le s \le r,$$

$$\gamma_{r,s}(\Lambda) = \gamma_{r,s} \cdot q^{-\sum\limits_{i=1}^{r} l_{i,s}} e_s\left(q^{2l_{1,r}}, q^{2l_{2,r}}, \dots, q^{2l_{r,r}}\right), \tag{5.4}$$

where $\gamma_{r,s} = [s]![r-s]!q^{-r^2+(r+1)s}$ and e_s are the elementary symmetric polinomials.

By analogy with the theorem 4.2 one can prove the following

Proposition 5.3. *Suppose that λ is an admissible top row. Then the action of Γ^N_q in the representation T_λ is diagonizable in the basis of modified Gelfand-Tsetlin tableaux:*

$$T_\lambda(c_{r,s})(\xi_\Lambda) = \gamma_{r,s}(\Lambda)\xi_\Lambda,\ \textit{for } r \le N,\ 0 \le s \le r.$$

Corollary 5.4. *Representations from series T^c_λ, T^s_λ and T^p_λ are pairwise non-equivalent for different values of parameters.*

Corollary 5.5. *The action of Γ^N_q separates tableaux in the representation T_λ. It means that for every two modified G-T tableax there exists an element of Γ^N_q that has non-equal eigenvalues on that tableaux.*

Proof. Suppose we have two tableaux $\Lambda_1, \Lambda_2 \in S_\lambda$ such that $\gamma_{r,s}(\Lambda_1) = \gamma_{r,s}(\Lambda_2)$ for all $r \le N,\ 0 \le s \le r$. Then by Proposition 5.3 we get that for all $r \le N$ the collections $\left(q^{2l^1_{1,r}}, q^{2l^1_{2,r}}, \dots, q^{2l^1_{r,r}}\right)$ and $\left(q^{2l^2_{1,r}}, q^{2l^2_{2,r}}, \dots, q^{2l^2_{r,r}}\right)$ differ by a permutation. That is possible iff this permutation is the identity because of the condition 3.1. □

Lemma 5.6. *Suppose that λ is an admissible top row and $W \subset V_\lambda$ is an invariant subspace. If $a_1\xi_{\Lambda_1}+, \dots, a_1\xi_{\Lambda_k} \in W$ for $a_1, \dots, a_k \neq 0$, then $\xi_{\Lambda_1}, \dots, \xi_{\Lambda_k} \in W$.*

Proposition 5.7. *Suppose that λ is an admissible top row. Then T_λ is an irreducible representation.*

Proof. If $W \subset V_\lambda$ is an invariant subspace, then it follows from Lemma 5.6 that the corresponding subrepresentation is a direct summand of T_λ. It is easy to see that one can obtain every tableau from V_λ by increasing or decreasing the value of elements in the tableau μ by an integer (see Figure 2).

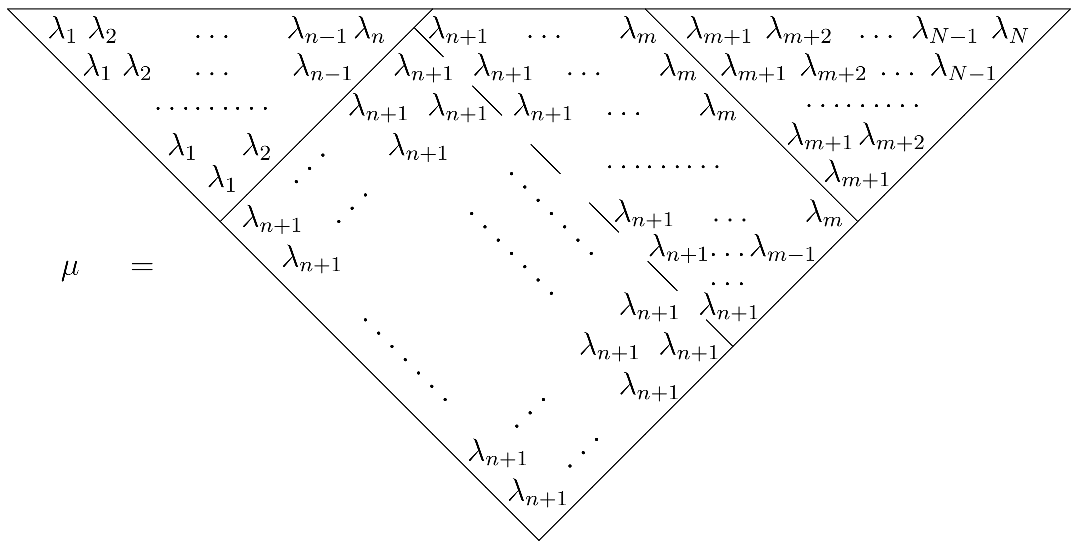

FIGURE 2

□

Corollary 5.8. *The following branching law holds*

$$\operatorname{Res}^N_{N-1} T_\lambda = \bigoplus_{\mu \prec \lambda} T_\mu,$$

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