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FONCTIONS HARMONIQUES SUR LES GRAPHES RAMIFIÉS

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Chapitre 1

Introduction en Français

Cette thèse porte sur les fonctions harmoniques semi-finies sur les graphes de branchement.

1.1 Fonctions harmoniques et graphes de branchement : premier exemple

L'exemple le plus simple d'un graphe de branchement est celui du graphe de Pascal $\mathbb{P}$, dont l'ensemble des sommets est $\mathbb{Z}^2_{\geq 0}$. Deux sommets de $\mathbb{P}$ sont reliés par une arête si et seulement si leurs premières coordonnées coïncident et que leurs deuxièmes coordonnées diffèrent de un, ou vice versa. Ainsi, il n'y a que deux arêtes sortant de chaque sommet de $\mathbb{P}$. Le graphe de Pascal est étroitement lié à l'algèbre des polynômes à deux variables $\mathbb{R}[x,y]$. En d'autres termes, on peut dire que les sommets de $\mathbb{P}$ sont étiquetés par les éléments de base $x^n y^m$ et que les arêtes reflètent la multiplication par $x+y$:

$$(x+y)x^n y^m = x^{n+1}y^m + x^n y^{m+1}.$$

Une fonction φ non-négative sur $\mathbb{P}$ est appelée *harmonique*, si

$$\varphi(n,m) = \varphi(n+1,m) + \varphi(n,m+1).$$

En d'autres termes, φ est cohérente avec le *branchement* de $\mathbb{P}$. Une telle fonction φ definit une forme liniaire non-négative (dans le sens où elle prend des valeurs non-négatives sur la base $x^n y^m$) sur $\mathbb{R}[x,y]$, qui s'annule sur l'idéal engendré par $(x+y-1) \subset \mathbb{R}[x,y]$. De plus, les fonctions harmoniques indécomposables correspondent à des formes liniaires *multiplicatives*, c'est-à-dire aux homomorphismes d'algèbres $\mathbb{R}[x,y] \to \mathbb{R}$.

Les fonctions harmoniques finies sur le graphe de Pascal sont en correspondance bijective avec les mesures de probabilité sur l'intervalle unité fermé. Ce résultat est équivalent au célèbre théorème de de Finetti. Cette équivalence découle du fait que les fonctions harmoniques finies sur $\mathbb{P}$ satisfaisant la condition $\varphi(0,0) = 1$ sont en correspondance bijective avec les mesures de probabilité dites *centrales* sur l'espace des chemins de $\mathbb{P}$, qui est l'espace des séquences infinies composées de 0 et de 1. La condition de centralité signifie que la mesure d'un cylindre ne dépend que du nombre de 0 et de 1 dans la partie initiale du chemin, mais pas de leur position.

1.2 Travaux antérieurs et motivation

La motivation pour étudier les fonctions harmoniques semi-finies vient de la théorie des algèbres d'opérateurs. La théorie classique des caractères des groupes finis et des groupes compacts peut être généralisée à d'autres classes de groupes et d'algèbres de différentes manières. Pour les groupes et les C^*-algèbres qui ne sont pas de type I, la théorie des caractères n'est pas liée aux représentations irréductibles mais aux représentations factorielles normales, c'est-à-dire aux homomorphismes d'algèbres de von Neumann avec une trace finie ou semi-finie. Pour les AF-algèbres, on peut reformuler la théorie des caractères dans un langage combinatoire-algébrique grâce aux fonctions harmoniques non-négatives sur les diagrammes de Bratteli. De manière équivalente, on peut traiter ces fonctions harmoniques comme des mesures centrales sur l'espace des chemins monotones dans le graphe. Cette approche a été développée dans les travaux d'A. M. Vershik et S. V. Kerov à la fin des années 70 - début des années 80. Les fonctions harmoniques qui prennent uniquement des valeurs finies conduisent à des mesures de probabilité sur l'espace des chemins. Ces fonctions sont en bijection avec les traces finies et correspondent aux représentations factorielles de type fini. L'analogie avec les représentations factorielles de types I_∞ et II_∞ suggère d'étudier les fonctions harmoniques dites semi-finies. La propriété de semi-finitude signifie que les fonctions peuvent prendre la valeur $+\infty$ et que ces valeurs infinies peuvent être approximées par des valeurs finies. Les fonctions harmoniques semi-finis sont en bijection avec les traces semi-finites inférieurement semi-continues sur des C^*-algèbres appropriées, voir le Théorème 1.9 et la Définition 1.8 de [4]. Ce fait montre que les fonctions harmoniques fournissent un cadre combinatoire approprié pour l'étude des problèmes de classification des traces sur les AF-algèbres.

A. M. Vershik et S. V. Kerov ont obtenu la classification des fonctions harmoniques semi-finies sur les graphes de Young et de Kingman, [12, 9]. Ils ont résolu ce problème à l'aide de la méthode ergodique, qui implique l'évaluation d'une limite hautement non triviale. En principe, cette méthode peut être appliquée à n'importe quel graphe de branchement, mais sa difficulté essentielle, qui n'est pas toujours facile à surmonter, est de calculer cette limite. Il existe une autre approche développée par A. J. Wassermann. Dans sa thèse [31], il a suggéré d'utiliser une bijection entre les représentations factorielles fidèles d'une C^*-algèbre primitive A et celles d'un idéal fermé bilatère arbitraire de A [31, p. 143, Théorème 7]. Il existe un autre ingrédient important de la méthode de Wassermann [31, p.146, Théorème 8]. Cette méthode nécessite que le groupe de Grothendieck de la C^*-algèbre en question admette une structure d'anneau qui est un domaine intégral satisfaisant certaines contraintes supplémentaires. A. Wassermann a appliqué sa méthode pour déterminer toutes les fonctions harmoniques semi-finies indécomposables sur le graphe de Young et a prouvé le résultat de classification de Vershik et Kerov [12, 9] en évitant la méthode ergodique ou tout autre calcul analytique compliqué.

1.3 Organisation de la thèse

Dans la section 3, nous développons une version combinatoire de la méthode de Wassermann, que nous utiliserons dans les sections suivantes. Dans la section 4, nous décrivons les fonctions harmoniques finies et semi-finies sur le produit direct de graphes de branchement en termes de telles fonctions sur les facteurs. Dans la section 5, nous décrivons les fonctions harmoniques semi-finies sur le graphe en zigzag et prouvons un analogue semi-fini du théorème de l'anneau de Vershik-Kerov pour celui-ci.

1.4 Résultats principaux

1.4.1 Version combinatoire de la méthode de Wassermann

La méthode de Wassermann est basée sur cinq énoncés dont nous présentons des analogues combinatoires ci-dessous, voir la Proposition 1.4.5, le Théorème 1.4.9, la Proposition 1.4.10, le Théorème 1.4.12 et la Proposition 1.4.13. Ces résultats seront utilisés dans les sections suivantes. Les énoncés originaux sont formulés et démontrés à l'aide de la théorie des algèbres d'opérateurs, tandis que nous les prouvons de manière purement combinatoire. Cela nous permet de simplifier et de clarifier considérablement les démonstrations. De plus, nous travaillons avec une généralisation des diagrammes de Bratteli - nous considérons des graphes de branchement avec des multiplicités formelles non-négatives sur les arêtes.

Définition 1.4.1. Par un *graphe gradué*, nous entendons une paire $(\Gamma, \varkappa)$, où Γ est un ensemble gradué $\Gamma = \bigsqcup_{n \geq 0} \Gamma_n$, Γ_n étant des ensembles finis et $\varkappa$ est une fonction $\Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$, qui satisfait aux contraintes suivantes :

1) si $\lambda \in \Gamma_n$ et $\mu \in \Gamma_m$, alors $\varkappa(\lambda, \mu) = 0$ pour tous $m - n \neq 1$.

2) pour tout sommet $\lambda \in \Gamma_n$, il existe $\mu \in \Gamma_{n+1}$ tel que $\varkappa(\lambda, \mu) \neq 0$.

Les arêtes du graphe gradué $(\Gamma, \varkappa)$ sont, par définition, des paires de sommets (λ, μ) avec $\varkappa(\lambda, \mu) > 0$. Nous pouvons donc considérer $\varkappa(\lambda, \mu)$ comme une multiplicité formelle de l'arête.

Si $\lambda \in \Gamma_n$, alors le nombre n est unique. Nous le notons par $|\lambda|$. Nous écrivons $\lambda \nearrow \mu$, si $|\mu| - |\lambda| = 1$ et $\varkappa(\lambda, \mu) \neq 0$. Dans ce cas, nous disons qu'il *existe une arête de λ à μ de multiplicité $\varkappa(\lambda, \mu)$*.

Soit $\mu, \nu \in \Gamma$ et $|\nu| - |\mu| = n \geq 1$. Alors l'expression suivante

$$\dim(\mu, \nu) = \sum_{\substack{\lambda_0, \ldots, \lambda_n \in \Gamma: \\ \mu = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_{n-1} \nearrow \lambda_n = \nu}} \varkappa(\lambda_0, \lambda_1)\varkappa(\lambda_1, \lambda_2) \ldots \varkappa(\lambda_{n-1}, \lambda_n).$$

représente le nombre pondéré de chemins de μ à ν. Par définition, nous avons également $\dim(\mu, \mu) = 1$ et $\dim(\mu, \nu) = 0$ si $\nu \not\geq \mu$.

Définition 1.4.2. Un *graphe de branchement* est défini comme un graphe gradué $(\Gamma, \varkappa)$ qui satisfait les conditions suivantes :

- $\Gamma_0 = \varnothing$ est un singleton,
- pour tout $\lambda \in \Gamma_n$ avec $n \geq 1$, il existe $\mu \in \Gamma_{n-1}$ tel que $\mu \nearrow \lambda$.

Soit Γ un graphe gradué.

Définition 1.4.3. Un sous-ensemble $J \subset \Gamma$ est appelé un *co-idéal* si pour tous les sommets $\lambda \in J$ et $\mu \in \Gamma$ tels que $\mu < \lambda$, on a $\mu \in J$.

Définition 1.4.4. Un co-idéal J est appelé *saturé* si pour tout $\lambda \in J$, il existe un sommet $\mu \in J$ tel que $\lambda \nearrow \mu$. Un co-idéal saturé J est appelé *primitif* si pour tous les co-idéaux saturés J_1 et J_2 tels que $J = J_1 \cup J_2$, on a $J = J_1$ ou $J = J_2$.

Soit Γ un graphe de branchement. L'espace des chemins infinis dans Γ commençant par $\varnothing$ sera noté $\mathcal{T}(\Gamma)$. À chaque chemin $\tau = (\varnothing \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \ldots) \in \mathcal{T}(\Gamma)$, nous associons le co-idéal saturé primitif $\Gamma_\tau = \bigcup_{n \geq 1} \{\lambda \in \Gamma \mid \lambda \leq \lambda_n\}$.

Proposition 1.4.5. 1) Un co-idéal saturé J d'un graphe gradué est primitif si et seulement si pour tout couple de sommets $\lambda_1, \lambda_2 \in J$, on peut trouver un sommet $\mu \in J$ tel que $\mu \geq \lambda_1, \lambda_2$.

2) Tout co-idéal primitif saturé d'un graphe de branchement est de la forme $J = \Gamma_\tau$ pour un certain chemin $\tau \in \mathcal{T}(\Gamma)$.

Définition 1.4.6. Un graphe gradué Γ est appelé *primitif* s'il est primitif en tant que co-idéal, c'est-à-dire que pour tout couple de sommets $\lambda_1, \lambda_2 \in \Gamma$, il existe un sommet $\mu \in \Gamma$ tel que $\mu \geq \lambda_1, \lambda_2$.

Définition 1.4.7. Soit $(\Gamma, \varkappa)$ un graphe gradué. Une fonction $\varphi\colon \Gamma \to \mathbb{R}_{\geq 0} \cup +\infty$ est appelée *harmonique* si elle vérifie la propriété suivante :

$$\varphi(\lambda) = \sum_{\mu:\lambda \nearrow \mu} \varkappa(\lambda, \mu)\varphi(\mu), \quad \forall \lambda \in \Gamma.$$

Nous convenons que

- $x + (+\infty) = +\infty$, pour tout $x \in \mathbb{R}$,
- $(+\infty) + (+\infty) = +\infty$,
- $0 \cdot (+\infty) = 0$.

Le symbole $\mathrm{K}_0(\Gamma)$ désigne le $\mathbb{R}$-espace vectoriel engendré par les sommets de Γ sujet aux relations suivantes

$$\lambda = \sum_{\mu:\lambda \nearrow \mu} \varkappa(\lambda, \mu) \cdot \mu, \quad \forall \lambda \in \Gamma.$$

Le symbole $\mathrm{K}_0^+(\Gamma)$ désigne le cône positif dans $\mathrm{K}_0(\Gamma)$, généré par les sommets de Γ, c'est-à-dire $\mathrm{K}_0^+(\Gamma) = \mathrm{span}_{\mathbb{R}_{\geq 0}}(\lambda \mid \lambda \in \Gamma)$. L'ordre partiel, défini par le cône $\mathrm{K}_0^+(\Gamma)$, est noté $\geq_K$. Cela signifie que $a \geq_K b \iff a - b \in \mathrm{K}_0^+(\Gamma)$. Par exemple, si $\lambda \leq \mu$, alors $\lambda \geq_K \dim(\lambda, \mu) \cdot \mu$.

L'application $\mathbb{R}_{\geq 0}$-linéaire $\mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup +\infty$, définie par une fonction harmonique φ, sera notée par le même symbole φ.

Définition 1.4.8. Une fonction harmonique φ est appelée *semi-finie*, si elle n'est pas finie et l'application $\varphi\colon\ \mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup +\infty$ vérifie la propriété suivante

$$\varphi(a) = \sup_{\substack{b \in \mathrm{K}_0^+(\Gamma):\ b \leq_K a, \\ \varphi(b) < +\infty}} \varphi(b), \qquad \forall a \in \mathrm{K}_0^+(\Gamma). \tag{1.1}$$

L'ensemble de toutes les fonctions harmoniques indécomposables finies (non identiquement nulles) et semi-finies sur un graphe gradué Γ est noté $\mathcal{H}_{\mathrm{ex}}(\Gamma)$. Le sous-ensemble de $\mathcal{H}_{\mathrm{ex}}(\Gamma)$ composé de fonctions strictement positives est noté $\mathcal{H}^\circ_{\mathrm{ex}}(\Gamma)$.

Théorème 1.4.9. Soit I un idéal d'un graphe gradué Γ.

1) Il existe une correspondance bijective entre $\{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\}$ et $\mathcal{H}_{\mathrm{ex}}(I)$, définie par les applications mutuellement inverses suivantes :

$$\mathrm{Res}_I^\Gamma : \{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\} \to \mathcal{H}_{\mathrm{ex}}(I), \;\; \varphi \mapsto \varphi\big|_I,$$
$$\mathrm{Ext}_I^\Gamma : \mathcal{H}_{\mathrm{ex}}(I) \to \{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\}, \;\; \varphi(\cdot) \mapsto \lim_{N\to\infty} \sum_{\substack{\mu \in I \\ |\mu| = N}} \dim(\cdot, \mu)\varphi(\mu).$$

2) Si Γ est un graphe gradué primitif, alors la bijection ci-dessus préserve strictement les fonctions harmoniques strictement positives $\mathcal{H}^\circ_{\mathrm{ex}}(I) \longleftrightarrow \mathcal{H}^\circ_{\mathrm{ex}}(\Gamma)$.

Proposition 1.4.10. Soit Γ un graphe gradué. Si $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)$, alors le support $\operatorname{supp}(\varphi) := \lambda \in \Gamma \mid \varphi(\lambda) > 0$ est un co-idéal primitif.

Définition 1.4.11. Un graphe de branchement Γ est appelé *multiplicatif* s'il existe une $\mathbb{R}$-algèbre associative graduée $\mathbb{Z}_{\geq 0}$, $A = \bigoplus_{n \geq 0} A_n$, $A_0 = \mathbb{R}$, avec une base distinguée d'éléments homogènes $a_\lambda \lambda \in \Gamma$ qui satisfont les conditions suivantes :

1) $\deg a_\lambda = |\lambda|$,

2) $a_\varnothing$ est l'élément identité dans A,

3) Pour $\widehat{a} = \sum_{\nu \in \Gamma_1} \varkappa(\varnothing, \nu) a_\nu$ et tout sommet $\lambda \in \Gamma$, nous avons $\widehat{a} \cdot a_\lambda = \sum_{\mu : \lambda \nearrow \mu} \varkappa(\lambda, \mu) a_\mu$.

De plus, nous supposons que les constantes de structure de A par rapport à la base $\{a_\lambda\}_{\lambda \in \Gamma}$ sont non-négatives.

Théorème 1.4.12 (Théorème de non-existence de Wassermann)**.** Soit Γ un graphe multiplicatif. Si $a_\lambda a_\mu \neq 0$ pour tous les $\lambda, \mu \in \Gamma$, alors le graphe Γ n'admet pas de fonctions harmoniques indécomposables strictement positives et semi-finies.

Proposition 1.4.13 (Lemme de Boyer)**.** Soit Γ un graphe gradué et φ une fonction harmonique sur ce graphe. Soit $I \subset \Gamma$ un idéal et $J = \Gamma \backslash I$ le co-idéal correspondant et $\lambda \in J$ un sommet fixé. Supposons qu'il existe un sommet $\lambda' \in I$ et un nombre réel positif β_λ tels que $\varphi(\lambda') > 0$ et que pour tout sommet $\eta \in I$ situé à un niveau suffisamment élevé, l'inégalité suivante soit vérifiée

$$\sum_{\mu \in J} \dim(\lambda, \mu) \varkappa(\mu, \eta) \geq \beta_\lambda \dim(\lambda', \eta).$$

Alors $\varphi(\lambda) = +\infty$. Si de plus (1.1) est vérifiée pour $a = \lambda'$, alors elle l'est également pour $a = \lambda$.

1.4.2 Produit direct de graphes ramifiés

Ici, nous décrivons les fonctions harmoniques finies et semi-finies sur le produit direct de graphes de branchement en termes de telles fonctions sur les facteurs.

Définition 1.4.14. Par *produit direct* de graphes gradués $(\Gamma_1, \varkappa_1)$ et $(\Gamma_2, \varkappa_2)$, nous entendons le graphe gradué $(\Gamma_1 \times \Gamma_2, \varkappa_1 \times \varkappa_2)$, où

$$(\Gamma_1 \times \Gamma_2)_k = \bigsqcup_{\substack{n,m \geq 0: \\ n+m=k}} (\Gamma_1)_n \times (\Gamma_2)_m$$

et

$$(\varkappa_1 \times \varkappa_2)\Big((\lambda_1, \mu_1); (\lambda_2, \mu_2)\Big) = \begin{cases} \varkappa_1(\lambda_1, \lambda_2), & \text{si } \mu_1 = \mu_2, \\ \varkappa_2(\mu_1, \mu_2), & \text{si } \lambda_1 = \lambda_2, \\ 0 & \text{sinon.} \end{cases}$$

Exemple 1.4.15. Le triangle de Pascal est le produit direct de deux copies de $\mathbb{Z}_{\geq 0}$.

Nous notons par $\mathcal{FH}_{\text{ex}}(\Gamma)$ l'ensemble de toutes les fonctions harmoniques finies normalisées indécomposables sur un graphe de branchement Γ.

Théorème 1.4.16. Soient Γ_1 et Γ_2 des graphes de branchement et φ une fonction harmonique finie normalisée indécomposable sur $\Gamma_1 \times \Gamma_2$, c'est-à-dire $\varphi \in \mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$. Alors un seul des cas suivants peut se produire :

1) Il existe $\varphi_1 \in \mathcal{FH}_{ex}(\Gamma_1)$, $\varphi_2 \in \mathcal{FH}_{ex}(\Gamma_2)$ et des nombres réels positifs w_1, w_2 tels que $w_1 + w_2 = 1$ et
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda)\varphi_2(\mu). \tag{1.2}$$
De plus, ces φ_1, φ_2, w_1, w_2 sont uniques.

2) Il existe $\varphi_1 \in \mathcal{FH}_{ex}(\Gamma_1)$ telle que
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \mu \neq \varnothing, \\ \varphi_1(\lambda), & \text{si } \mu = \varnothing. \end{cases} \tag{1.3}$$

3) Il existe $\varphi_2 \in \mathcal{FH}_{ex}(\Gamma_2)$ telle que
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \lambda \neq \varnothing, \\ \varphi_2(\mu), & \text{si } \lambda = \varnothing. \end{cases} \tag{1.4}$$

De plus, chaque fonction harmonique sur $\Gamma_1 \times \Gamma_2$ de la forme 1), 2) ou 3) est indécomposable.

Remarque 1.4.17. On peut facilement voir que (1.3) et (1.4) sont des cas particuliers de (1.2) correspondant à $w_2 = 0$ et $w_1 = 0$. Nous formulons le Théorème 1.4.16 sous cette forme pour simplifier la comparaison avec le Théorème 1.4.18.

Théorème 1.4.18. Soient Γ_1 et Γ_2 des graphes gradués et $\varphi \in \mathcal{H}_{ex}(\Gamma_1 \times \Gamma_2)$, alors un seul des cas suivants peut se produire :

1) Il existe $\varphi_1 \in \mathcal{H}_{ex}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}_{ex}(\Gamma_2)$ et des nombres réels positifs w_1, w_2 avec $w_1 + w_2 = 1$ tels que
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda)\varphi_2(\mu).$$
De plus, ces φ_1 et φ_2 sont définis de manière unique à une constante multiplicative près.

2) Il existe $\varphi_1 \in \mathcal{H}_{ex}(\Gamma_1)$ et $\nu_2 \in \Gamma_2$ tels que
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \mu \not\leq \nu_2, \\ +\infty, & \text{si } \mu < \nu_2, \\ \varphi_1(\lambda), & \text{si } \mu = \nu_2. \end{cases}$$

3) Il existe $\nu_1 \in \Gamma_1$ et $\varphi_2 \in \mathcal{H}_{ex}(\Gamma_2)$ tels que
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \lambda \not\leq \nu_1, \\ +\infty, & \text{si } \lambda < \nu_1, \\ \varphi_2(\mu), & \text{si } \lambda = \nu_1. \end{cases}$$

De plus, chaque fonction harmonique sur $\Gamma_1 \times \Gamma_2$ de la forme 1), 2) ou 3) est finie ou semi-finie et indécomposable.

1.4.3 Le graphe en zigzag

Dans cette section, nous décrivons les fonctions harmoniques semi-finies sur le graphe en zigzag et prouvons un analogue semi-fini du théorème de l'anneau de Vershik-Kerov pour celui-ci.

Considérons les compositions (partitions ordonnées) de nombres naturels. Nous les identifions avec les diagrammes en ruban, qui sont des diagrammes de Young imbriqués connectés écrits selon la convention française et ne contenant pas de blocs de boîtes de taille 2×2. Une composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ est identifiée avec le diagramme de Young en ruban ayant λ_i boîtes dans la i-ème lignes. Par exemple, la seule composition de 1 est identifiée avec $\square$. Le nombre de boîtes dans λ est égal à $|\lambda| = \lambda_1 + \ldots + \lambda_l$. Nous considérons les diagrammes de Young en ruban comme des zigzags se déplaçant du coin supérieur gauche au coin inférieur droit. Il existe une bijection entre les zigzags et les mots binaires.

Un mot binaire est un mot dans l'alphabet de deux symboles, + et −. Nous utiliserons les conventions suivantes

$$\overset{n}{+} = \underbrace{+\ldots+}_{n} \quad \text{et} \quad \overset{n}{-} = \underbrace{-\ldots-}_{n}.$$

La bijection entre les zigzags et les mots binaires est la suivante. De gauche à droite, nous lisons les symboles du mot binaire et ajoutons des cases au zigzag le plus simple $\square$. Si le symbole est +, nous ajoutons une case dans la direction horizontale vers la droite, et si le symbole est −, nous ajoutons une case dans la direction verticale vers le bas. Par exemple, le mot binaire $-+$ correspond au zigzag avec une case dans la première rangée et deux cases dans la deuxième rangée. Le mot binaire correspondant à un zigzag λ sera noté $\mathsf{bw}(\lambda)$. Ainsi, $\mathsf{bw}(\square)$ est le mot binaire vide.

Chaque mot binaire peut être représenté de manière unique comme une réunion consécutive de *blocs* avec des signes alternés. Par un *bloc*, nous entendons un uplet de symboles du même signe. Par exemple, le mot $+-\overset{3}{+}$ se divise en trois blocs, +, − et $\overset{3}{+}$. Ainsi, un bloc peut être positif ou négatif en fonction du signe des symboles. En ce qui concerne les zigzags, ces blocs positifs et négatifs correspondent aux lignes et aux colonnes.

Par un *cluster*, nous entendons un symbole, + ou −, auquel est associée une multiplicité positive formelle, qui peut être infinie. Nous disons qu'un cluster est *infini*, si sa multiplicité est infinie, sinon nous disons que le cluster est *fini*. Un *modèle* est une collection ordonnée de clusters alternés. De plus, nous supposons toujours qu'un modèle contient au moins un cluster infini.

Définition 1.4.19. Un modèle est appelé *fini*, s'il *ne contient pas* de clusters finis, à l'exception de ceux qui ne sont pas extrémaux et dont les deux voisins sont des clusters infinis du même signe. Un modèle qui n'est pas fini sera appelé *semi-fini*.

Soit t un modèle semi-fini. Par un *cluster séparant* de t, nous entendons un cluster d'un seul symbole qui n'est pas un cluster extrémal de t et dont les deux voisins sont des clusters infinis du même signe. Par le *flanc zigzag* de t, nous entendons un uplet de mots binaires dont chaque élément est composé de clusters finis mais non séparants de t se tenant à proximité. Le flanc zigzag sera noté $\mathsf{fl}(t)$.

Soit t un modèle semi-fini. Par une *section* de t, nous entendons une collection maximale de clusters consécutifs formant un modèle fini. Remarquons que les mots du flanc zigzag de t divisent t en sections.

Soit t un modèle arbitraire. Par t_n, nous désignons le mot binaire qui est obtenu à partir de t en remplaçant toutes les multiplicités infinies par le nombre naturel n. Alors le sous-ensemble du graphe en zigzag $\mathcal{Z}(t) := \{\lambda \in \mathcal{Z} \mid \mathsf{bw}(\lambda) < t_n \text{ pour un certain } n\}$ consiste en tous les zigzags (ou mots binaires) qui sont de *la forme* t.

Définition 1.4.20. Posons $J(t) = \bigcup_r \mathbb{Z}(r)$ pour un modèle semi-fini t, où la réunion est prise sur tous les r obtenus à partir de t en supprimant un seul symbole d'un cluster correspondant à un bloc d'un mot binaire du flanc zigzag $\mathsf{fl}(t)$.

Supposons que t ait k sections $t_1, \dots, t_k$. Supposons que $\mathsf{fl}(t) = (a_0, \dots, a_k)$ et que la partition de t soit de la forme :

$$t = (a_0,\ t_1,\ a_1,\ \dots\ , a_{k-1},\ t_k,\ a_k).$$

Si a_0 ou a_k est le mot binaire vide, nous l'ignorons dans tout ce qui suit.

Lemme 1.4.21. Si $\lambda \in \mathbb{Z}(t) \backslash J(t)$, alors

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k$$

pour certains $\lambda^{(i)} \in \mathbb{Z}(t_i)$, qui sont alors uniques.

Définition 1.4.22. Par un *modèle de croissance en zigzag semi-fini*, nous entendons une paire (t, w), où t est un gabarit semi-fini ayant m grappes infinies et $w = (w_1, \dots, w_m)$ est un m-uplet de nombres réels positifs tels que $w_1 + \ldots + w_m = 1$.

Soit (t, w) un modèle de croissance en zigzag semi-fini. La partition de t en sections nous donne une partition de w

$$w = v_1 \sqcup \ldots \sqcup v_k,$$

où chaque v_i est un uplet de nombres réels provenant de $w = (w_1, \dots, w_m)$ correspondant aux clusters infinis de t_i.

Définition 1.4.23. Pour tout $\lambda \in \mathbb{Z}$, nous définissons

$$\varphi_{t,w}(\lambda) = \begin{cases} F_{\lambda^{(1)}}(v_1) \cdot \ldots \cdot F_{\lambda^{(k)}}(v_k), \text{ si } \lambda \in \mathbb{Z}(t) \backslash J(t), \\ +\infty, \text{ si } \lambda \in J(t), \\ 0, \text{ si } \lambda \notin \mathbb{Z}(t). \end{cases}$$

où $\lambda \mapsto (\lambda^{(1)}, \dots, \lambda^{(k)})$ est l'application donnée par le Lemme 1.4.21 et $F_{\lambda^{(i)}}(v_i)$ est défini comme suit :

$$F_\mu(x_1, x_2, \dots, x_n) = \sum x_1^{|\mu(1)|} x_2^{|\mu(2)|} \ldots x_n^{|\mu(n)|}, \tag{1.5}$$

où la somme est prise sur les décompositions de μ en n zigzags $\mu(1), \dots, \mu(n)$ tels que $\mu(i)$ soit une ligne, si le nombre $x_i \in \{w_1, \dots, w_m\}$ correspond à un cluster positif de t, et $\mu(i)$ soit une colonne, si x_i correspond à un cluster négatif de t. Notons que certains de ces $\mu(i)$ peuvent être vides.

Notous que l'expression (1.5) provient d'une application multiplicative $QSym \to \mathbb{R}$, voir la Section 5.4.1 sur la construction de Kerov.

Théorème 1.4.24.

1) Pour tout modèle de croissance en zigzag semi-fini (t, w), la fonction $\varphi_{t,w}$ est une fonction harmonique semi-finie indécomposable sur $\mathbb{Z}$.

2) Toute fonction harmonique semi-finie indécomposable sur $\mathbb{Z}$ est proportionnelle à $\varphi_{t,w}$ pour certains modèles de croissance en zigzag semi-finis (t, w).[1]

[1] Notous que l'ensemble des zéros de $\varphi_{t,w}$ est toujours non vide. Cela est conforme au théorème de non-existence de Wassermann en raison du fait que le graphe en zigzag est multiplicatif et que $QSym$ ne contient aucun diviseur de zéro, voir le Théorème 1.4.12.

3) Les fonctions $\varphi_{t,w}$ sont distinctes pour des modèles de croissance en zigzag semi-finis distincts (t, w).

Maintenant, nous aimerions formuler le théorème de l'anneau de Vershik-Kerov semi-fini pour le graphe en zigzag. Pour cela, nous étendons nos fonctions harmoniques semi-finis sur $\mathcal{Z}$ à $\operatorname{span}_{\mathbb{R}_{\geq 0}}\{F_\lambda \mid \lambda \in \mathcal{Z}\} \subset QSym$, où $\{F_\lambda\}_{\lambda \in \mathcal{Z}}$ sont les fonctions quasi-symétriques fondamentales.

Théorème 1.4.25. Soit (t, w) un modèle de croissance en zigzag semi-fini. Pour tout $\mu \in \mathcal{Z}(t) \backslash J(t)$ et $\lambda \in \mathcal{Z}$, nous avons

$$\varphi_{t,w}\left(F_\lambda F_\mu\right) = \varphi_w(F_\lambda)\varphi_{t,w}(F_\mu),$$

où φ_w est la fonction harmonique finie sur $\mathcal{Z}$ définie par $\varphi_w(\lambda) = F_\lambda(w)$, voir la formule (1.5) ci-dessus.

1.5 Publications et conférences

La thèse est basée sur trois articles :

1. N. A. Safonkin. "Semifinite harmonic functions on branching graphs". In : *Journal of Mathematical Sciences (New York)* 261 (2022), p. 669-686. arXiv : 2108.07850 [math.RT]

2. P. Nikitin et N. Safonkin. "Semifinite harmonic functions on the direct product of graded graphs". In : *Representation theory, dynamical systems, combinatorial methods. Part XXXIV, Zap. Nauchn. Sem. POMI* 517 (2022), p. 125-150

3. N. Safonkin. "Semifinite harmonic functions on the zigzag graph". In : *Functional Analysis and Its Applications* 56 :3 (2022), p. 52-74. arXiv : 2110.01508 [math.RT]

Exposés basés sur les résultats de la présente thèse :

1. 2023, *Semifinite harmonic functions on the zigzag graph*, Les Probas du vendredi, LPSM, Sorbonne University, January 27, link.

2. 2022, *Graded graphs and related topics*, Working Seminar on Mathematical Physics of HSE and Skoltech CAS, September 21, link.

3. 2021, *Wassermann's method for the Young and zigzag graphs*, St. Petersburg Seminar on Representation Theory and Dynamical Systems, PDMI, December 22, link.

4. 2021, *Semifinite harmonic functions on the Young and zigzag graphs*, St. Petersburg Seminar on Representation Theory and Dynamical Systems, PDMI, December 8, link.

5. 2021, *Semifinite harmonic functions*, Representations and Probability seminar, HSE, March 29, link.

6. 2021, *Harmonic functions on the zigzag graph*, Representations and Probability seminar, HSE, March 22, link.

7. 2021, *The zigzag graph and the path space of a branching graph*, Representations and Probability seminar, HSE, March 15, link.

8. 2020, Central and invariant measures and applications, August 17-21, 2020, Euler International Mathematical Institute, St. Petersburg, Russia.

1.6 Remerciements

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Chapter 2

Introduction in English

My PhD thesis concerns semifinite harmonic functions on branching graphs.

2.1 Harmonic functions and branching graphs: first example

The simplest example of a branching graph is the Pascal graph $\mathbb{P}$, which is $\mathbb{Z}^2_{\geq 0}$. Two vertices of $\mathbb{P}$ are joined by an edge if and only if either their first coordinates coincide and the second coordinates differ by one, or vice versa. So, there are only two edges going out of every vertex of $\mathbb{P}$. The Pascal graph is closely related to the algebra of polynomials in two variables $\mathbb{R}[x,y]$. Namely, one can say that the vertices of $\mathbb{P}$ are labelled by the basis elements $x^n y^m$ and the edges reflect the multiplication by $x+y$:

$$(x+y)x^n y^m = x^{n+1} y^m + x^n y^{m+1}.$$

A non-negative function φ on $\mathbb{P}$ is called *harmonic*, if

$$\varphi(n,m) = \varphi(n+1,m) + \varphi(n,m+1).$$

In other words, φ is consistent with the *branching* of $\mathbb{P}$. Such φ gives us a non-negative (in the sense that it takes non-negative values on the basis $x^n y^m$) functional on $\mathbb{R}[x,y]$, which vanishes on the ideal $(x+y-1) \subset \mathbb{R}[x,y]$. Furthermore, indecomposable harmonic functions correspond to *multiplicative* functionals, i.e. algebra homomorphisms $\mathbb{R}[x,y] \to \mathbb{R}$.

Finite harmonic functions on the Pascal graph are in a bijective correspondence with probability measures on the closed unit interval. This result is equivalent to the celebrated de Finetti theorem. This equivalence follows from the fact that the finite harmonic functions on $\mathbb{P}$ satisfying the condition $\varphi(0,0) = 1$ are in a bijective correspondence with the so-called *central* probability measures on the path space of $\mathbb{P}$, which is the space of infinite sequences comprised of 0's and 1's. The centrality condition means that the measure of a cylinder depends only on the number of 0's and 1's in the initial part of the path, but not on their positions.

2.2 Earlier works and motivation

The motivation to study semifinite harmonic functions comes from the theory of operator algebras. Classical character theory of finite and compact groups may be generalized to other classes of groups and algebras in various ways. For groups and C^*-algebras *not* of type I the character theory is related *not* to irreducible representations but to normal factor representations, i.e. homomorphisms to von Neumann algebras with a finite or semifinite trace.

For AF-algebras one can reformulate the character theory in a combinatorial-algebraic language, speaking about non-negative harmonic functions on Bratteli diagrams. Equivalently, one can treat these harmonic functions as central measures on the space of monotone paths in the graph. This approach was developed in works of A. M. Vershik and S. V. Kerov in the late of 70's — early 80's. Harmonic functions that take only finite values lead to probability measures on the path space. These functions are in bijection with finite traces and correspond to factor representations of finite types. Analogy with factor representations of types I_∞ and II_∞ motivates us to study the so-called semifinite harmonic functions. The semifiniteness property means that the functions may take the value $+\infty$ and these infinite values can be approximated by finite ones. Semifinite harmonic functions are in bijection with semifinite lower-semicontinuous traces on appropriate C^*-algebras, see Theorem 1.9 and Definition 1.8 from [4]. This fact indicates that harmonic functions provide a suitable combinatorial framework for classification problems of traces on AF-algebras.

A. M. Vershik and S. V. Kerov have obtained the classification of semifinite harmonic functions on the Young and Kingman graphs, [12, 9]. They solved this problem with the help of the so-called ergodic method, which involves evaluation of a very non-trivial limit. In principle this method can be applied to any branching graph, but the main difficulty, which is not always easy to overcome, is to compute that limit. There is another approach developed by A. J. Wassermann. In his dissertation [31], he suggested to use a bijection between faithfull factor representations of a primitive C^*-algebra A and those of an arbitrary closed two-sided ideal of A [31, p. 143, Theorem 7]. There is another ingredient of Wassermann's method [31, p.146, Theorem 8]. It requires that the Grothendieck group of the C^*-algebra in question admits a ring structure which turns it into an integral domain satisfying some additional constraints. A. Wassermann applied his method to determine all indecomposable semifinite harmonic functions on the Young graph and proved the classification result of Vershik and Kerov [12, 9] without the ergodic method or any other complicated analytical computations.

2.3 Organisation of the thesis

In Section 3 we develop a combinatorial version of Wassermann's method, which we will use in further sections. In Section 4 we describe finite and semifinite harmonic functions on the direct product of branching graphs in terms of the same functions on the factors. In Section 5 we describe semifinite harmonic functions on the zigzag graph and prove a semifinite analog of the Vershik-Kerov ring theorem for it.

2.4 Main results

2.4.1 Combinatorial version of Wassermann's method

Wassermann's method is based on five statements combinatorial analogs of which are presented below, see Proposition 2.4.5, Theorem 2.4.9, Proposition 2.4.10, Theorem 2.4.12, Proposition 2.4.13. These results will be used in the next sections. The original claims are formulated and proved with the help of the operator algebras theory, while we prove them in a purely combinatorial way. This allows us to simplify and clarify the proofs significantly. Moreover, we work with a generalisation of Bratteli diagrams – we consider branching graphs with formal non-negative multiplicities on edges.

Definition 2.4.1. By a *graded graph* we mean a pair $(\Gamma, \varkappa)$, where Γ is a graded set $\Gamma = \bigsqcup\limits_{n\geq 0} \Gamma_n$, Γ_n are finite sets and $\varkappa$ is a function $\Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$, that satisfies the following constraints:

1) if $\lambda \in \Gamma_n$ and $\mu \in \Gamma_m$, then $\varkappa(\lambda, \mu) = 0$ for $m - n \neq 1$.

2) for any vertex $\lambda \in \Gamma_n$ there exists $\mu \in \Gamma_{n+1}$ with $\varkappa(\lambda, \mu) \neq 0$.

Edges of the graded graph $(\Gamma, \varkappa)$ are, by definition, pairs of vertices (λ, μ) with $\varkappa(\lambda, \mu) > 0$. Then we may treat $\varkappa(\lambda, \mu)$ as a formal multiplicity of the edge.

If $\lambda \in \Gamma_n$, then the number n is uniquely defined. We denote it by $|\lambda|$. We write $\lambda \nearrow \mu$, if $|\mu| - |\lambda| = 1$ and $\varkappa(\lambda, \mu) \neq 0$. In this case we say that *there is an edge from λ to μ of multiplicity $\varkappa(\lambda, \mu)$.*

Let $\mu, \nu \in \Gamma$ and $|\nu| - |\mu| = n \geq 1$. Then the following expression

$$\dim(\mu, \nu) = \sum_{\substack{\lambda_0, \ldots, \lambda_n \in \Gamma: \\ \mu = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_{n-1} \nearrow \lambda_n = \nu}} \varkappa(\lambda_0, \lambda_1)\varkappa(\lambda_1, \lambda_2) \ldots \varkappa(\lambda_{n-1}, \lambda_n).$$

is the "weighted" number of paths from μ to ν. By definition we also set $\dim(\mu, \mu) = 1$ and $\dim(\mu, \nu) = 0$, if $\nu \not\geq \mu$.

Definition 2.4.2. A *branching graph* is defined as a graded graph $(\Gamma, \varkappa)$ that satisfies the following conditions

- $\Gamma_0 = \{\varnothing\}$ is a singleton,
- for any $\lambda \in \Gamma_n$ with $n \geq 1$ there exists $\mu \in \Gamma_{n-1}$ such that $\mu \nearrow \lambda$.

Let Γ be a graded graph.

Definition 2.4.3. A subset $J \subset \Gamma$ is called a *coideal*, if for any vertices $\lambda \in J$ and $\mu \in \Gamma$ such that $\mu < \lambda$ we have $\mu \in J$.

Definition 2.4.4. A coideal J is called *saturated*, if for any $\lambda \in J$ there exists a vertex $\mu \in J$ such that $\lambda \nearrow \mu$. A saturated coideal J is called *primitive*, if for any saturated coideals J_1, J_2 such that $J = J_1 \cup J_2$ we have $J = J_1$ or $J = J_2$.

Let Γ be a branching graph. The space of infinite paths in Γ starting at $\varnothing$ will be denoted by $\mathcal{T}(\Gamma)$. To every path $\tau = (\varnothing \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \ldots) \in \mathcal{T}(\Gamma)$ we associate the saturated primitive coideal $\Gamma_\tau = \bigcup\limits_{n \geq 1} \{\lambda \in \Gamma \mid \lambda \leq \lambda_n\}$.

Proposition 2.4.5. *1) A saturated coideal J of a graded graph is primitive if and only if for any two vertices $\lambda_1, \lambda_2 \in J$ we can find a vertex $\mu \in J$ such that $\mu \geq \lambda_1, \lambda_2$.*

2) Every saturated primitive coideal of a branching graph is of the form $J = \Gamma_\tau$ for some path $\tau \in \mathcal{T}(\Gamma)$.

Definition 2.4.6. A graded graph Γ is called *primitive* if it is primitive as a coideal, i.e. for any vertices $\lambda_1, \lambda_2 \in \Gamma$ there exists a vertex $\mu \in \Gamma$ such that $\mu \geq \lambda_1, \lambda_2$.

Definition 2.4.7. Let $(\Gamma, \varkappa)$ be a graded graph. A function $\varphi\colon \Gamma \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is called *harmonic*, if it enjoys the following property:

$$\varphi(\lambda) = \sum_{\mu: \lambda \nearrow \mu} \varkappa(\lambda, \mu)\varphi(\mu), \quad \forall \lambda \in \Gamma.$$

We agree that

- $x + (+\infty) = +\infty$, for any $x \in \mathbb{R}$,

- $(+\infty)+(+\infty)=+\infty$,
- $0\cdot(+\infty)=0$.

The symbol $\mathrm{K}_0(\Gamma)$ stands for the $\mathbb{R}$-vector space spanned by the vertices of Γ subject to the following relations

$$\lambda=\sum_{\mu:\lambda\nearrow\mu}\varkappa(\lambda,\mu)\cdot\mu,\quad \forall\lambda\in\Gamma.$$

The symbol $\mathrm{K}_0^+(\Gamma)$ denotes the positive cone in $\mathrm{K}_0(\Gamma)$, generated by the vertices of Γ, i.e. $\mathrm{K}_0^+(\Gamma)=\mathrm{span}_{\mathbb{R}_{\geq 0}}(\lambda\mid\lambda\in\Gamma)$. The partial order, defined by the cone $\mathrm{K}_0^+(\Gamma)$, is denoted by $\geq_K$. That is $a\geq_K b \iff a-b\in\mathrm{K}_0^+(\Gamma)$. For instance, if $\lambda\leq\mu$, then $\lambda\geq_K \dim(\lambda,\mu)\cdot\mu$.

The $\mathbb{R}_{\geq 0}$-linear map $\mathrm{K}_0^+(\Gamma)\to\mathbb{R}_{\geq 0}\cup\{+\infty\}$, defined by a harmonic function φ, will be denoted by the same symbol φ.

Definition 2.4.8. A harmonic function φ is called *semifinite*, if it is not finite and the map $\varphi\colon \mathrm{K}_0^+(\Gamma)\to\mathbb{R}_{\geq 0}\cup\{+\infty\}$ enjoys the following property

$$\varphi(a)=\sup_{\substack{b\in\mathrm{K}_0^+(\Gamma)\colon\, b\leq_K a,\\ \varphi(b)<+\infty}}\varphi(b),\qquad \forall a\in\mathrm{K}_0^+(\Gamma). \tag{2.1}$$

The set of all indecomposable finite (not identically zero) and semifinite harmonic functions on a graded graph Γ is denoted by $\mathcal{H}_{\mathrm{ex}}(\Gamma)$. The subset of $\mathcal{H}_{\mathrm{ex}}(\Gamma)$ consisting of strictly positive functions is denoted by $\mathcal{H}^{\circ}_{\mathrm{ex}}(\Gamma)$.

Theorem 2.4.9. *Let I be an ideal of a graded graph Γ.*

1) There is a bijective correspondence between $\{\varphi\in\mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I\neq 0\}$ and $\mathcal{H}_{\mathrm{ex}}(I)$, defined by the following mutually inverse maps

$$\begin{aligned}&\mathrm{Res}^{\Gamma}_I:\{\varphi\in\mathcal{H}_{\mathrm{ex}}(\Gamma)\colon\varphi|_I\neq 0\}\to\mathcal{H}_{\mathrm{ex}}(I),\;\;\varphi\mapsto\varphi\big|_I,\\ &\mathrm{Ext}^{\Gamma}_I:\mathcal{H}_{\mathrm{ex}}(I)\to\{\varphi\in\mathcal{H}_{\mathrm{ex}}(\Gamma)\colon\varphi|_I\neq 0\},\;\;\varphi(\cdot)\mapsto\lim_{N\to\infty}\sum_{\substack{\mu\in I\\|\mu|=N}}\dim(\cdot,\mu)\varphi(\mu).\end{aligned}$$

2) If Γ is a primitive graded graph, then the bijection above preserves strictly positive harmonic functions $\mathcal{H}^{\circ}_{\mathrm{ex}}(I)\longleftrightarrow\mathcal{H}^{\circ}_{\mathrm{ex}}(\Gamma)$.

Proposition 2.4.10. *Let Γ be a graded graph. If $\varphi\in\mathcal{H}_{\mathrm{ex}}(\Gamma)$, then the support* $\mathrm{supp}(\varphi):=\{\lambda\in\Gamma\mid\varphi(\lambda)>0\}$ *is a primitive coideal.*

Definition 2.4.11. A branching graph Γ is called *multiplicative*, if there exists an associative $\mathbb{Z}_{\geq 0}$-graded $\mathbb{R}$-algebra $A=\bigoplus_{n\geq 0}A_n$, $A_0=\mathbb{R}$ with a distinguished basis of homogeneous elements $\{a_\lambda\}_{\lambda\in\Gamma}$, that satisfy the following conditions

1) $\deg a_\lambda=|\lambda|$
2) $a_{\varnothing}$ is the identity in A
3) For $\widehat{a}=\sum_{\nu\in\Gamma_1}\varkappa(\varnothing,\nu)a_\nu$ and any vertex $\lambda\in\Gamma$ we have $\widehat{a}\cdot a_\lambda=\sum_{\mu:\lambda\nearrow\mu}\varkappa(\lambda,\mu)a_\mu$.

Moreover, we assume that the structure constants of A with respect to the basis $\{a_\lambda\}_{\lambda\in\Gamma}$ are non-negative.

Theorem 2.4.12 (Wassermann's no-go theorem)**.** *Let Γ be a multiplicative graph. If $a_\lambda a_\mu \neq 0$ for any $\lambda, \mu \in \Gamma$, then the graph Γ admits no strictly positive semifinite indecomposable harmonic functions.*

Proposition 2.4.13 (Boyer's lemma)**.** *Let Γ be a graded graph and φ be a harmonic function on it. Assume that $I \subset \Gamma$ is an ideal, $J = \Gamma \backslash I$ is the corresponding coideal and we are given a fixed vertex $\lambda \in J$. Suppose that there exists a vertex $\lambda' \in I$ and a positive real number β_λ such that $\varphi(\lambda') > 0$ and for any vertex $\eta \in I$ lying on a large enough level the following inequality holds*

$$\sum_{\mu \in J} \dim(\lambda, \mu) \varkappa(\mu, \eta) \geq \beta_\lambda \dim(\lambda', \eta).$$

Then $\varphi(\lambda) = +\infty$. If in addition (2.1) *holds for $a = \lambda'$, then it holds for $a = \lambda$ as well.*

2.4.2 Direct product of branching graphs

Here we describe finite and semifinite harmonic functions on the direct product of branching graphs in terms of the same functions on the factors.

Definition 2.4.14. By the *direct product* of graded graphs $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$ we mean the graded graph $(\Gamma_1 \times \Gamma_2, \varkappa_1 \times \varkappa_2)$, where

$$(\Gamma_1 \times \Gamma_2)_k = \bigsqcup_{\substack{n,m \geq 0: \\ n+m=k}} (\Gamma_1)_n \times (\Gamma_2)_m$$

and

$$(\varkappa_1 \times \varkappa_2)\Big((\lambda_1, \mu_1); (\lambda_2, \mu_2)\Big) = \begin{cases} \varkappa_1(\lambda_1, \lambda_2), & \text{if } \mu_1 = \mu_2, \\ \varkappa_2(\mu_1, \mu_2), & \text{if } \lambda_1 = \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.4.15. The Pascal triangle is the direct product of two copies of $\mathbb{Z}_{\geq 0}$.

We denote by $\mathcal{FH}_{\text{ex}}(\Gamma)$ the set of all finite normalized indecomposable harmonic functions on a branching graph Γ.

Theorem 2.4.16. *Let Γ_1 and Γ_2 be branching graphs and φ be a normalized indecomposable finite harmonic function on $\Gamma_1 \times \Gamma_2$, i.e. $\varphi \in \mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$. Then only one of the following situations can occur:*

1) *There exist $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$ and real positive numbers w_1, w_2 with $w_1 + w_2 = 1$ such that*
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda) \varphi_2(\mu). \tag{2.2}$$
Moreover, these φ_1, φ_2, w_1, w_2 are uniquely defined.

2) *There exist $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \mu \neq \varnothing, \\ \varphi_1(\lambda), & \text{if } \mu = \varnothing. \end{cases} \tag{2.3}$$

3) *There exist $\varphi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \lambda \neq \varnothing, \\ \varphi_2(\mu), & \text{if } \lambda = \varnothing. \end{cases} \tag{2.4}$$

Furthermore, every harmonic function on $\Gamma_1 \times \Gamma_2$ of the form 1), 2), or 3) is indecomposable.

Remark 2.4.17. One can readily see that (2.3) and (2.4) are partial cases of (2.2) corresponding to $w_2 = 0$ and $w_1 = 0$. We formulate Theorem 2.4.16 in this form to simplify the comparison with Theorem 2.4.18.

Theorem 2.4.18. *Let Γ_1 and Γ_2 be graded graphs and $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$, then only one of the following situations can occur:*

1) *There exist $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ and real positive numbers w_1, w_2 with $w_1 + w_2 = 1$ such that*
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda)\varphi_2(\mu).$$
Moreover, these φ_1 and φ_2 are defined uniquely up to multiplicative constants.
We agree that $0 \cdot (+\infty) = 0$.

2) *There exist $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$ and $\nu_2 \in \Gamma_2$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \mu \not\leq \nu_2, \\ +\infty, & \text{if } \mu < \nu_2, \\ \varphi_1(\lambda), & \text{if } \mu = \nu_2. \end{cases}$$

3) *There exist $\nu_1 \in \Gamma_1$ and $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \lambda \not\leq \nu_1, \\ +\infty, & \text{if } \lambda < \nu_1, \\ \varphi_2(\mu), & \text{if } \lambda = \nu_1. \end{cases}$$

Furthermore, every harmonic function on $\Gamma_1 \times \Gamma_2$ of the form 1), 2), or 3) is finite or semifinite, and indecomposable.

2.4.3 The zigzag graph

In this section we describe semifinite harmonic functions on the zigzag graph and prove a semifinite analog of the Vershik-Kerov ring theorem for it.

Let us consider compositions (ordered partitions) of natural numbers. We identify them with the ribbon diagrams, which are connected skew Young diagrams written in the French notation and containing no 2×2 blocks of boxes. A composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is identified with the ribbon Young diagram having λ_i boxes in the i-th row. For instance, the only one composition of 1 gets identified with $\square$. The number of boxes in λ equals $|\lambda| = \lambda_1 + \ldots + \lambda_l$. We treat ribbon Young diagrams as zigzags crawling from the top-left corner to the bottom-right corner. There is a bijection between the zigzags and the binary words.

A binary word is a word in the alphabet of two symbols, $+$ and $-$. We will use the following conventions
$$\overset{n}{+} = \underbrace{+\ldots+}_{n} \quad \text{and} \quad \overset{n}{-} = \underbrace{-\ldots-}_{n}.$$

The bijection between the zigzags and the binary words is as follows. From left to right we read the symbols off the binary word and add boxes to the simplest zigzag $\square$. If the symbol is $+$, then we add a box in the horizontal direction to the right, and if the symbol is $-$, then we add a box in the vertical direction to the bottom. For instance, the binary word $-+$ corresponds to the zigzag with one box in the first row and two boxes in the second row.

The binary word corresponding to a zigzag λ will be denoted by $\mathsf{bw}(\lambda)$. So, $\mathsf{bw}(\square)$ is the empty binary word.

Each binary word can be uniquely represented as a consecutive union of *blocks* with alternating signs. By a *block* we mean a tuple of symbols of the same sign. For instance, the word $+-\overset{3}{+}$ splits into three blocks, $+$, $-$, and $\overset{3}{+}$. So, a block can be positive or negative depending on the sign of symbols. As for zigzags, these positive and negative blocks correspond to rows and columns.

By a *cluster* we mean a symbol, $+$ or $-$, with an assigned to it formal positive multiplicity, which may be infinite. We say that a cluster is *infinite*, if its multiplicity is infinite, otherwise we say that the cluster is *finite*. A *template* is an ordered collection of alternating clusters. Furthermore, we always assume that a template contains at least one infinite cluster.

Definition 2.4.19. A template is called *finite*, if it *does not* contain finite clusters except those one-symbol clusters which are not outermost and whose two neighbors are infinite clusters of the same sign. A template which is not finite will be called *semifinite*.

Let t be a semifinite template. By a *separating cluster* of t we mean a one-symbol cluster which is not an outermost cluster of t and whose two neighbors are infinite clusters of the same sign. By the *zigzag flange* of t we call a tuple of binary words each of which consists of finite but not separating clusters of t standing nearby. The zigzag flange will be denoted by $\mathsf{fl}(t)$.

Let t be a semifinite template. By a *section* of t we mean a maximal collection of consecutive clusters that form a finite template. Note that the words from the zigzag flange of t split t into sections.

Let t be an arbitrary template. By t_n we denote the binary word which is obtained from t by replacing all infnite multiplicities by the natural number n. Then the subset of the zigzag graph $\mathbb{Z}(t) := \{\lambda \in \mathbb{Z} \mid \mathsf{bw}(\lambda) < t_n \text{ for some } n\}$ consists of all the zigzags (or binary words) that are of *the form t*.

Definition 2.4.20. Let us set $J(t) = \bigcup_r \mathbb{Z}(r)$ for a semifinite template t, where the union is taken over all r obtained from t by removing a single symbol from some cluster corresponding to a block of a binary word from the zigzag flange $\mathsf{fl}(t)$.

Let t have k sections $t_1,\dots,t_k$. Assume that $\mathsf{fl}(t) = (a_0,\dots,a_k)$ and the splitting of t into sections looks like

$$t = (a_0,\ t_1,\ a_1,\ \dots\ ,a_{k-1},\ t_k,\ a_k).$$

If a_0 or a_k is the empty binary word, then we should merely ignore it in all what follows.

Lemma 2.4.21. *If $\lambda \in \mathbb{Z}(t)\backslash J(t)$, then*

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \dots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k$$

for some $\lambda^{(i)} \in \mathbb{Z}(t_i)$, which are uniquely defined.

Definition 2.4.22. By a *semifinite zigzag growth model* we call a pair (t,w), where t is a semifinite template having m infinite clusters and $w = (w_1,\dots,w_m)$ is an m-tuple of positive real numbers such that $w_1 + \dots + w_m = 1$.

Let (t,w) be a semifinite zigzag growth model. The splitting of t into sections gives us a splitting of w

$$w = v_1 \sqcup \dots \sqcup v_k,$$

where each v_i is a tuple of real numbers from $w = (w_1,\dots,w_m)$ corresponding to the infinite clusters of t_i.

Definition 2.4.23. For any $\lambda \in \mathbb{Z}$ we set

$$\varphi_{t,w}(\lambda) = \begin{cases} F_{\lambda^{(1)}}(v_1) \cdot \ldots \cdot F_{\lambda^{(k)}}(v_k), \text{ if } \lambda \in \mathbb{Z}(t) \backslash J(t), \\ +\infty, \text{ if } \lambda \in J(t), \\ 0, \text{ if } \lambda \notin \mathbb{Z}(t). \end{cases}$$

where $\lambda \mapsto (\lambda^{(1)}, \ldots, \lambda^{(k)})$ is the map provided by Lemma 2.4.21 and $F_{\lambda^{(i)}}(v_i)$ is defined as follows:

$$F_\mu(x_1, x_2, \ldots, x_n) = \sum x_1^{|\mu(1)|} x_2^{|\mu(2)|} \ldots x_n^{|\mu(n)|}, \tag{2.5}$$

where the sum is taken over the following splittings of μ into n zigzags $\mu(1), \ldots, \mu(n)$ such that $\mu(i)$ is a row, if the number $x_i \in \{w_1, \ldots, w_m\}$ corresponds to a positive cluster of t, and $\mu(i)$ is a column, if x_i corresponds to a negative cluster of t. Note that some of these $\mu(i)$ may be empty.

Note that expression (2.5) comes from a multiplicative map $QSym \to \mathbb{R}$, see Section 5.4.1 on Kerov's construction.

Theorem 2.4.24.

1) *For any semifinite zigzag growth model (t, w) the function $\varphi_{t,w}$ is a semifinite indecomposable harmonic function on $\mathbb{Z}$.*

2) *Any semifinite indecomposable harmonic function on $\mathbb{Z}$ is proportional to $\varphi_{t,w}$ for some semifinite zigzag growth model (t, w).*[1]

3) *The functions $\varphi_{t,w}$ are distinct for distinct semifinite zigzag growth models (t, w).*

Now we would like to formulate the semifinite Vershik-Kerov ring theorem for the zigzag graph. For that we extend our semifinite harmonic functions on $\mathbb{Z}$ to $\mathrm{span}_{\mathbb{R}_{\geq 0}}(F_\lambda \mid \lambda \in \mathbb{Z}) \subset QSym$, where $\{F_\lambda\}_{\lambda \in \mathbb{Z}}$ are fundamental quasisymmetric functions.

Theorem 2.4.25. *Let (t, w) be a semifinite zigzag growth model. For any $\mu \in \mathbb{Z}(t) \backslash J(t)$ and $\lambda \in \mathbb{Z}$ we have*

$$\varphi_{t,w}\left(F_\lambda F_\mu\right) = \varphi_w(F_\lambda) \varphi_{t,w}(F_\mu),$$

where φ_w is the finite harmonic function on $\mathbb{Z}$ defined by $\varphi_w(\lambda) = F_\lambda(w)$, see formula (2.5) above.

2.5 Publications and Talks

The thesis is based on the three papers:

1. N. A. Safonkin. "Semifinite harmonic functions on branching graphs". In: *Journal of Mathematical Sciences (New York)* 261 (2022), pp. 669–686. arXiv: `2108.07850 [math.RT]`

2. P. Nikitin and N. Safonkin. "Semifinite harmonic functions on the direct product of graded graphs". In: *Representation theory, dynamical systems, combinatorial methods. Part XXXIV, Zap. Nauchn. Sem. POMI* 517 (2022), pp. 125–150

3. N. Safonkin. "Semifinite harmonic functions on the zigzag graph". In: *Functional Analysis and Its Applications* 56:3 (2022), pp. 52–74. arXiv: `2110.01508 [math.RT]`

[1] Note that the zero set of $\varphi_{t,w}$ is always non-empty. This is in agreement with Wassermann's no-go theorem due to the fact that the zigzag graph is multiplicative and $QSym$ contains no zero divisors, see Theorem 2.4.12.

Talks based on the results of the present thesis:

1. 2023, *Semifinite harmonic functions on the zigzag graph*, Les Probas du vendredi, LPSM, Sorbonne University, January 27, link.

2. 2022, *Graded graphs and related topics*, Working Seminar on Mathematical Physics of HSE and Skoltech CAS, September 21, link.

3. 2021, *Wassermann's method for the Young and zigzag graphs*, St. Petersburg Seminar on Representation Theory and Dynamical Systems, PDMI, December 22, link.

4. 2021, *Semifinite harmonic functions on the Young and zigzag graphs*, St. Petersburg Seminar on Representation Theory and Dynamical Systems, PDMI, December 8, link.

5. 2021, *Semifinite harmonic functions*, Representations and Probability seminar, HSE, March 29, link.

6. 2021, *Harmonic functions on the zigzag graph*, Representations and Probability seminar, HSE, March 22, link.

7. 2021, *The zigzag graph and the path space of a branching graph*, Representations and Probability seminar, HSE, March 15, link.

8. 2020, Central and invariant measures and applications, August 17-21, 2020, Euler International Mathematical Institute, St. Petersburg, Russia.

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Chapter 3

Combinatorial version of Wassermann's method

3.1 Summary in French

La méthode de Wassermann est basée sur cinq énoncés dont nous présentons des analogues combinatoires ci-dessous, voir la Proposition 3.1.5, le Théorème 3.1.9, la Proposition 3.1.10, le Théorème 3.1.12 et la Proposition 3.1.13. Ces résultats seront utilisés dans les sections suivantes. Les énoncés originaux sont formulés et démontrés à l'aide de la théorie des algèbres d'opérateurs, tandis que nous les prouvons de manière purement combinatoire. Cela nous permet de simplifier et de clarifier considérablement les démonstrations. De plus, nous travaillons avec une généralisation des diagrammes de Bratteli - nous considérons des graphes de branchement avec des multiplicités formelles non-négatives sur les arêtes.

Définition 3.1.1. Par un *graphe gradué*, nous entendons une paire $(\Gamma, \varkappa)$, où Γ est un ensemble gradué $\Gamma = \bigsqcup_{n\geq 0} \Gamma_n$, Γ_n étant des ensembles finis et $\varkappa$ est une fonction $\Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$, qui satisfait aux contraintes suivantes:

1) si $\lambda \in \Gamma_n$ et $\mu \in \Gamma_m$, alors $\varkappa(\lambda, \mu) = 0$ pour tous $m - n \neq 1$.

2) pour tout sommet $\lambda \in \Gamma_n$, il existe $\mu \in \Gamma_{n+1}$ tel que $\varkappa(\lambda, \mu) \neq 0$.

Les arêtes du graphe gradué $(\Gamma, \varkappa)$ sont, par définition, des paires de sommets (λ, μ) avec $\varkappa(\lambda, \mu) > 0$. Nous pouvons donc considérer $\varkappa(\lambda, \mu)$ comme une multiplicité formelle de l'arête.

Si $\lambda \in \Gamma_n$, alors le nombre n est unique. Nous le notons par $|\lambda|$. Nous écrivons $\lambda \nearrow \mu$, si $|\mu| - |\lambda| = 1$ et $\varkappa(\lambda, \mu) \neq 0$. Dans ce cas, nous disons qu'il *existe une arête de λ à μ de multiplicité $\varkappa(\lambda, \mu)$*.

Soit $\mu, \nu \in \Gamma$ et $|\nu| - |\mu| = n \geq 1$. Alors l'expression suivante

$$\dim(\mu, \nu) = \sum_{\substack{\lambda_0, \dots, \lambda_n \in \Gamma: \\ \mu = \lambda_0 \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_{n-1} \nearrow \lambda_n = \nu}} \varkappa(\lambda_0, \lambda_1)\varkappa(\lambda_1, \lambda_2) \dots \varkappa(\lambda_{n-1}, \lambda_n).$$

représente le nombre pondéré de chemins de μ à ν. Par définition, nous avons également $\dim(\mu, \mu) = 1$ et $\dim(\mu, \nu) = 0$ si $\nu \not\geq \mu$.

Définition 3.1.2. Un *graphe de branchement* est défini comme un graphe gradué $(\Gamma, \varkappa)$ qui satisfait les conditions suivantes :

- $\Gamma_0 = \varnothing$ est un singleton,

- pour tout $\lambda \in \Gamma_n$ avec $n \geq 1$, il existe $\mu \in \Gamma_{n-1}$ tel que $\mu \nearrow \lambda$.

Soit Γ un graphe gradué.

Définition 3.1.3. Un sous-ensemble $J \subset \Gamma$ est appelé un *co-idéal* si pour tous les sommets $\lambda \in J$ et $\mu \in \Gamma$ tels que $\mu < \lambda$, on a $\mu \in J$.

Définition 3.1.4. Un co-idéal J est appelé *saturé* si pour tout $\lambda \in J$, il existe un sommet $\mu \in J$ tel que $\lambda \nearrow \mu$. Un co-idéal saturé J est appelé *primitif* si pour tous les co-idéaux saturés J_1 et J_2 tels que $J = J_1 \cup J_2$, on a $J = J_1$ ou $J = J_2$.

Soit Γ un graphe de branchement. L'espace des chemins infinis dans Γ commençant par $\varnothing$ sera noté $\mathcal{T}(\Gamma)$. À chaque chemin $\tau = (\varnothing \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \ldots) \in \mathcal{T}(\Gamma)$, nous associons le co-idéal saturé primitif $\Gamma_\tau = \bigcup\limits_{n \geq 1} \{\lambda \in \Gamma \mid \lambda \leq \lambda_n\}$.

Proposition 3.1.5.
1) Un co-idéal saturé J d'un graphe gradué est primitif si et seulement si pour tout couple de sommets $\lambda_1, \lambda_2 \in J$, on peut trouver un sommet $\mu \in J$ tel que $\mu \geq \lambda_1, \lambda_2$.
2) Tout co-idéal primitif saturé d'un graphe de branchement est de la forme $J = \Gamma_\tau$ pour un certain chemin $\tau \in \mathcal{T}(\Gamma)$.

Définition 3.1.6. Un graphe gradué Γ est appelé *primitif* s'il est primitif en tant que co-idéal, c'est-à-dire que pour tout couple de sommets $\lambda_1, \lambda_2 \in \Gamma$, il existe un sommet $\mu \in \Gamma$ tel que $\mu \geq \lambda_1, \lambda_2$.

Définition 3.1.7. Soit $(\Gamma, \varkappa)$ un graphe gradué. Une fonction $\varphi \colon \Gamma \to \mathbb{R}_{\geq 0} \cup +\infty$ est appelée *harmonique* si elle vérifie la propriété suivante:

$$\varphi(\lambda) = \sum_{\mu : \lambda \nearrow \mu} \varkappa(\lambda, \mu) \varphi(\mu), \quad \forall \lambda \in \Gamma.$$

Nous convenons que

- $x + (+\infty) = +\infty$, pour tout $x \in \mathbb{R}$,
- $(+\infty) + (+\infty) = +\infty$,
- $0 \cdot (+\infty) = 0$.

Le symbole $\mathrm{K}_0(\Gamma)$ désigne le $\mathbb{R}$-espace vectoriel engendré par les sommets de Γ sujet aux relations suivantes

$$\lambda = \sum_{\mu : \lambda \nearrow \mu} \varkappa(\lambda, \mu) \cdot \mu, \quad \forall \lambda \in \Gamma.$$

Le symbole $\mathrm{K}_0^+(\Gamma)$ désigne le cône positif dans $\mathrm{K}_0(\Gamma)$, généré par les sommets de Γ, c'est-à-dire $\mathrm{K}_0^+(\Gamma) = \mathrm{span}_{\mathbb{R}_{\geq 0}}(\lambda \mid \lambda \in \Gamma)$. L'ordre partiel, défini par le cône $\mathrm{K}_0^+(\Gamma)$, est noté $\geq_K$. Cela signifie que $a \geq_K b \iff a - b \in \mathrm{K}_0^+(\Gamma)$. Par exemple, si $\lambda \leq \mu$, alors $\lambda \geq_K \dim(\lambda, \mu) \cdot \mu$.

L'application $\mathbb{R}_{\geq 0}$-linéaire $\mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup +\infty$, définie par une fonction harmonique φ, sera notée par le même symbole φ.

Définition 3.1.8. Une fonction harmonique φ est appelée *semi-finie*, si elle n'est pas finie et l'application $\varphi\colon \mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup +\infty$ vérifie la propriété suivante

$$\varphi(a) = \sup_{\substack{b \in \mathrm{K}_0^+(\Gamma) \colon b \leq_K a, \\ \varphi(b) < +\infty}} \varphi(b), \qquad \forall a \in \mathrm{K}_0^+(\Gamma). \tag{3.1}$$

L'ensemble de toutes les fonctions harmoniques indécomposables finies (non identiquement nulles) et semi-finies sur un graphe gradué Γ est noté $\mathcal{H}_{\mathrm{ex}}(\Gamma)$. Le sous-ensemble de $\mathcal{H}_{\mathrm{ex}}(\Gamma)$ composé de fonctions strictement positives est noté $\mathcal{H}^{\circ}_{\mathrm{ex}}(\Gamma)$.

Théorème 3.1.9. Soit I un idéal d'un graphe gradué Γ.

1) Il existe une correspondance bijective entre $\{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\}$ et $\mathcal{H}_{\mathrm{ex}}(I)$, définie par les applications mutuellement inverses suivantes :

$$\mathrm{Res}^{\Gamma}_{I} : \{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\} \to \mathcal{H}_{\mathrm{ex}}(I), \;\; \varphi \mapsto \varphi\Big|_I,$$
$$\mathrm{Ext}^{\Gamma}_{I} : \mathcal{H}_{\mathrm{ex}}(I) \to \{\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)\colon \varphi|_I \neq 0\}, \;\; \varphi(\cdot) \mapsto \lim_{N\to\infty} \sum_{\substack{\mu\in I\\ |\mu|=N}} \dim(\cdot,\mu)\varphi(\mu).$$

2) Si Γ est un graphe gradué primitif, alors la bijection ci-dessus préserve strictement les fonctions harmoniques strictement positives $\mathcal{H}^{\circ}_{\mathrm{ex}}(I) \longleftrightarrow \mathcal{H}^{\circ}_{\mathrm{ex}}(\Gamma)$.

Proposition 3.1.10. Soit Γ un graphe gradué. Si $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$, alors le support $\mathrm{supp}(\varphi) := \lambda \in \Gamma \mid \varphi(\lambda) > 0$ est un co-idéal primitif.

Définition 3.1.11. Un graphe de branchement Γ est appelé *multiplicatif* s'il existe une $\mathbb{R}$-algèbre associative graduée $\mathbb{Z}_{\geq 0}$, $A = \bigoplus_{n\geq 0} A_n$, $A_0 = \mathbb{R}$, avec une base distinguée d'éléments homogènes $a_\lambda \lambda \in \Gamma$ qui satisfont les conditions suivantes :

1) $\deg a_\lambda = |\lambda|$,

2) $a_\varnothing$ est l'élément identité dans A,

3) Pour $\widehat{a} = \sum_{\nu\in\Gamma_1} \varkappa(\varnothing,\nu)a_\nu$ et tout sommet $\lambda \in \Gamma$, nous avons $\widehat{a}\cdot a_\lambda = \sum_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)a_\mu$.

De plus, nous supposons que les constantes de structure de A par rapport à la base $\{a_\lambda\}_{\lambda\in\Gamma}$ sont non-négatives.

Théorème 3.1.12 (Théorème de non-existence de Wassermann)**.** Soit Γ un graphe multiplicatif. Si $a_\lambda a_\mu \neq 0$ pour tous les $\lambda,\mu \in \Gamma$, alors le graphe Γ n'admet pas de fonctions harmoniques indécomposables strictement positives et semi-finies.

Proposition 3.1.13 (Lemme de Boyer)**.** Soit Γ un graphe gradué et φ une fonction harmonique sur ce graphe. Soit $I \subset \Gamma$ un idéal et $J = \Gamma\backslash I$ le co-idéal correspondant et $\lambda \in J$ un sommet fixé. Supposons qu'il existe un sommet $\lambda' \in I$ et un nombre réel positif β_λ tels que $\varphi(\lambda') > 0$ et que pour tout sommet $\eta \in I$ situé à un niveau suffisamment élevé, l'inégalité suivante soit vérifiée

$$\sum_{\mu\in J} \dim(\lambda,\mu)\varkappa(\mu,\eta) \geq \beta_\lambda \dim(\lambda',\eta).$$

Alors $\varphi(\lambda) = +\infty$. Si de plus (3.1) est vérifiée pour $a = \lambda'$, alors elle l'est également pour $a = \lambda$.

3.2 Ideals and coideals of graded graphs

In this section we recall main notions on branching graphs, ideals and coideals.

Definition 3.2.1. By a *graded graph* we mean a pair $(\Gamma,\varkappa)$, where Γ is a graded set $\Gamma = \bigsqcup_{n\geq 0} \Gamma_n$, Γ_n are finite sets and $\varkappa$ is a function $\Gamma\times\Gamma \to \mathbb{R}_{\geq 0}$, that satisfies the following constraints:

1) if $\lambda \in \Gamma_n$ and $\mu \in \Gamma_m$, then $\varkappa(\lambda, \mu) = 0$ for $m - n \neq 1$.

2) for any vertex $\lambda \in \Gamma_n$ there exists $\mu \in \Gamma_{n+1}$ with $\varkappa(\lambda, \mu) \neq 0$.

Edges of the graded graph $(\Gamma, \varkappa)$ are, by definition, pairs of vertices (λ, μ) with $\varkappa(\lambda, \mu) > 0$. Then we may treat $\varkappa(\lambda, \mu)$ as a formal multiplicity of the edge.

If $\lambda \in \Gamma_n$, then the number n is uniquely defined. We denote it by $|\lambda|$. We write $\lambda \nearrow \mu$, if $|\mu| - |\lambda| = 1$ and $\varkappa(\lambda, \mu) \neq 0$. In this case we say that *there is an edge from λ to μ of multiplicity $\varkappa(\lambda, \mu)$*.

Condition 1) from Definition 3.2.1 means that we allow edges only between adjacent levels and condition 2) means that each vertex must be connected by an edge with some vertex from the higher level.

A path in a graded graph Γ is a (finite or infinite) sequence of vertices $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $\lambda_i \nearrow \lambda_{i+1}$ for every i. We will write $\nu > \mu$ if $|\nu| > |\mu|$ and there is a path that connects μ and ν. We write $\nu \geq \mu$, if $\nu = \mu$ or $\nu > \mu$. Relation $\geq$ turns Γ into a poset.

Let $\mu, \nu \in \Gamma$ and $|\nu| - |\mu| = n \geq 1$. Then the following expression

$$\dim(\mu, \nu) = \sum_{\substack{\lambda_0, \ldots, \lambda_n \in \Gamma: \\ \mu = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_{n-1} \nearrow \lambda_n = \nu}} \varkappa(\lambda_0, \lambda_1) \varkappa(\lambda_1, \lambda_2) \ldots \varkappa(\lambda_{n-1}, \lambda_n). \tag{3.2}$$

is the "weighted" number of paths from μ to ν. By definition we also set $\dim(\mu, \mu) = 1$ and $\dim(\mu, \nu) = 0$, if $\nu \not\geq \mu$. The function $\dim(\cdot, \cdot) \colon \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ is called *the shifted dimension*. Note that $\dim(\mu, \nu) = \varkappa(\mu, \nu)$, if $\mu \nearrow \nu$, and for $\mu \in \Gamma_m$, $\nu \in \Gamma_n$ and any k such that $m \leq k \leq n$ we have

$$\dim(\mu, \nu) = \sum_{\lambda : \lambda \in \Gamma_k} \dim(\mu, \lambda) \dim(\lambda, \nu). \tag{3.3}$$

Definition 3.2.2. A *branching graph* is defined as a graded graph $(\Gamma, \varkappa)$ that satisfies the following conditions

- $\Gamma_0 = \{\varnothing\}$ is a singleton,
- for any $\lambda \in \Gamma_n$ with $n \geq 1$ there exists $\mu \in \Gamma_{n-1}$ such that $\mu \nearrow \lambda$.

For a branching graph $(\Gamma, \varkappa)$ we denote the expression $\dim(\varnothing, \lambda)$ by $\dim(\lambda)$ and call it *the dimension of λ*.

Definition 3.2.3. A subset of vertices I of a graded graph Γ is called an *ideal*, if for any vertices $\lambda \in I$ and $\mu \in \Gamma$ such that $\mu > \lambda$ we have $\mu \in I$. A subset $J \subset \Gamma$ is called a *coideal*, if for any vertices $\lambda \in J$ and $\mu \in \Gamma$ such that $\mu < \lambda$ we have $\mu \in J$.

Remark 3.2.4. Our terminology differs from the terminology of poset theory. Namely, our ideals and coideals are usually called *filters* and *ideals* respectively [26].

There is a bijective correspondence $I \leftrightarrow \Gamma \backslash I$ between ideals and coideals. Let J be a coideal and $I = \Gamma \backslash J$ be the corresponding ideal. Then the following conditions are equivalent:

1) if $\{\mu \mid \lambda \nearrow \mu\} \subset I$, then $\lambda \in I$

2) for any $\lambda \in J$ there exists a vertex $\mu \in J$ such that $\lambda \nearrow \mu$.

Definition 3.2.5. An ideal I and the corresponding coideal J are called *saturated*, if they satisfy the conditions above. A saturated ideal I is called *primitive*, if for any saturated ideals I_1, I_2 such that $I = I_1 \cap I_2$ we have $I = I_1$ or $I = I_2$. A saturated coideal J is called *primitive*, if for any saturated coideals J_1, J_2 such that $J = J_1 \cup J_2$ we have $J = J_1$ or $J = J_2$.

The bijection $I \leftrightarrow \Gamma \backslash I$ maps primitive saturated ideals to primitive saturated coideals and vice versa. We will also use the fact that ideals and saturated coideals are graded graphs themselves.

Let Γ be a branching graph. The space of infinite paths in Γ starting at $\varnothing$ will be denoted by $\mathcal{T}(\Gamma)$. To every path $\tau = (\varnothing \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \ldots) \in \mathcal{T}(\Gamma)$ we associate the saturated primitive coideal $\Gamma_\tau = \bigcup\limits_{n \geq 1} \{\lambda \in \Gamma \mid \lambda \leq \lambda_n\}$.

In the next proposition we give a combinatorial characterization of saturated primitive coideals of an arbitrary graded graph, see [5]. Moreover, for branching graphs we describe all such coideals in terms of path coideals Γ_τ, see [27] and [31, p.129].

Proposition 3.2.6. *1) A saturated coideal J of a graded graph is primitive if and only if for any two vertices $\lambda_1, \lambda_2 \in J$ we can find a vertex $\mu \in J$ such that $\mu \geq \lambda_1, \lambda_2$.*

2) Every saturated primitive coideal of a branching graph is of the form $J = \Gamma_\tau$ for some path $\tau \in \mathcal{T}(\Gamma)$.

Proof. Let $J \subset \Gamma$ be a saturated coideal. Suppose that there exist vertices $\lambda_1, \lambda_2 \in J$, that do not possess a common majorant. Let us prove that J may be presented as a union of two distinct proper saturated coideals. We need to introduce some notation. For any $\lambda \in J$ the subset of vertices of J, that lie above λ, will be denoted by J^λ, i.e. $J^\lambda = \{\mu \in J \mid \mu \geq \lambda\}$. For any subset $A \subset J$ we define $\downarrow A$ as the subset of vertices of J, that lie below some vertex of A, i.e. $\downarrow A = \{\mu \in J \mid \mu \leq \lambda$, for some $\lambda \in A\}$. Finally, for any ideal I of J the symbol $\operatorname{sat}(I)$ stands for the minimal saturated ideal that contains I. In other words, $\operatorname{sat}(I)$ consists of all the vertices of I and all vertices $\lambda \in J$ such that $\{\mu \,\big|\, \lambda \nearrow \mu\} \subset I$. With this notation in mind we set $J_1 = \downarrow(J^{\lambda_1}), J_2 = J \backslash \operatorname{sat}(J^{\lambda_1})$. It is not difficult to see that J_1 and J_2 are saturated coideals and their union coincides with J. Obviously, $\lambda_1 \in J_1$ and $\lambda_1 \notin J_2$. Next, we use the fact that vertices λ_1 and λ_2 do not possess a common majorant to show that $\lambda_2 \in J_2$ and $\lambda_2 \notin J_1$. Thus, J_1 and J_2 are proper distinct coideals of J.

Now suppose that for any vertices $\lambda_1, \lambda_2 \in J$ there exists $\mu \in J$ with $\mu \geq \lambda_1, \lambda_2$. We will show that $J = \Gamma_\tau$ for some path $\tau \in \mathcal{T}(\Gamma)$. Let us denote by $x_1, x_2, \ldots$ all the vertices of J enumerated in any (fixed) order. Since J is primitive, it follows that we can construct a sequence of vertices $y_1 \leq y_2 \leq \ldots$ of J with the following properties

$$\begin{array}{cccccc} & y_2 \geq y_1, & y_3 \geq y_2, & \ldots & y_n \geq y_{n-1} & \ldots \\ y_1 = x_1, & y_2 \geq x_2, & y_3 \geq x_3, & \ldots & y_n \geq x_n & \ldots \\ & y_2 \in J, & y_3 \in J, & \ldots & y_n \in J, & \ldots \end{array}$$

Let $\tau \in \mathcal{T}(\Gamma)$ be any path that goes through the vertices $y_1, y_2, \ldots$. Obviously, $J = \Gamma_\tau$. $\square$

Remark 3.2.7. One can formulate an obvious analog of the second part of Proposition 3.2.6 for arbitrary graded graphs, but this is of no particular importance to us.

Definition 3.2.8. A graded graph Γ is called *primitive* if it is primitive as a coideal, i.e. for any vertices $\lambda_1, \lambda_2 \in \Gamma$ there exists a vertex $\mu \in \Gamma$ such that $\mu \geq \lambda_1, \lambda_2$.

3.3 Semifinite harmonic functions

Definition 3.3.1. Let $(\Gamma,\varkappa)$ be a graded graph. A function $\varphi\colon \Gamma \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is called *harmonic*, if it enjoys the following property:

$$\varphi(\lambda) = \sum_{\mu:\lambda \nearrow \mu} \varkappa(\lambda,\mu)\varphi(\mu), \quad \forall \lambda \in \Gamma.$$

Throughout the paper we use the following conventions:

- $x + (+\infty) = +\infty$, for any $x \in \mathbb{R}$,
- $(+\infty) + (+\infty) = +\infty$,
- $0 \cdot (+\infty) = 0$.

Definition 3.3.2. The set of all vertices $\lambda \in \Gamma$ with $\varphi(\lambda) < +\infty$ is called the *finiteness ideal* of φ. We denote the *zero ideal* $\{\lambda \in \Gamma \mid \varphi(\lambda) = 0\}$ by $\ker\varphi$ and the *support* $\{\lambda \in \Gamma \mid \varphi(\lambda) > 0\}$ by $\operatorname{supp}\varphi$.

Note that the zero set $\ker(\varphi)$ is a saturated ideal and $\operatorname{supp}(\varphi)$ is a saturated coideal of Γ and $\ker(\varphi) \cup \operatorname{supp}(\varphi) = \Gamma$. Furthermore, we can restrict φ to any ideal or saturated coideal that contains $\operatorname{supp}(\varphi)$. The restriction is a harmonic function on that ideal or coideal respectively.

The symbol $\mathrm{K}_0(\Gamma)$ stands for the $\mathbb{R}$-vector space spanned by the vertices of Γ subject to the following relations

$$\lambda = \sum_{\mu:\lambda \nearrow \mu} \varkappa(\lambda,\mu) \cdot \mu, \quad \forall \lambda \in \Gamma.$$

The symbol $\mathrm{K}_0^+(\Gamma)$ denotes the positive cone in $\mathrm{K}_0(\Gamma)$, generated by the vertices of Γ, i.e. $\mathrm{K}_0^+(\Gamma) = \operatorname{span}_{\mathbb{R}_{\geq 0}}(\lambda \mid \lambda \in \Gamma)$. The partial order, defined by the cone $\mathrm{K}_0^+(\Gamma)$, is denoted by $\geq_K$. That is $a \geq_K b \iff a - b \in \mathrm{K}_0^+(\Gamma)$. For instance, if $\lambda \leq \mu$, then $\lambda \geq_K \dim(\lambda,\mu) \cdot \mu$.

Remark 3.3.3. Notation $\mathrm{K}_0(\Gamma)$ is motivated by the following fact. If all formal multiplicities of edges are integer numbers, then the vector space $\mathrm{K}_0(\Gamma)$ can be identified with the Grothendieck K_0-group of the corresponding AF-algebra. Under such a bijection the cone $\mathrm{K}_0^+(\Gamma)$ gets identified with the cone of true modules [14, Theorem 13 on page 32].

Observation 3.3.4. If $b \in \mathrm{K}_0^+(\Gamma)$ and $b \leq_K \lambda$ then b has the form

$$b = \sum_{\mu\colon |\mu| = N} b_\mu \mu$$

for some N and some real numbers b_μ subject to the following constraints $0 \leq b_\mu \leq \dim(\lambda,\mu)$. In particular, $b_\mu = 0$, if $\mu \ngeq \lambda$.

The $\mathbb{R}_{\geq 0}$-linear map $\mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$, defined by a harmonic function φ, will be denoted by the same symbol φ. Note that this map is monotone in the sense of the partial order. Namely, if $a \geq_K b$, then $\varphi(a) \geq \varphi(b)$.

Definition 3.3.5. A harmonic function φ is called *semifinite*, if it is not finite and the map $\varphi\colon \mathrm{K}_0^+(\Gamma) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ enjoys the following property

$$\varphi(a) = \sup_{\substack{b \in \mathrm{K}_0^+(\Gamma)\colon b \leq_K a, \\ \varphi(b) < +\infty}} \varphi(b), \qquad \forall a \in \mathrm{K}_0^+(\Gamma). \tag{3.4}$$

If $\varphi(a) < +\infty$, then condition (3.4) becomes the trivial identity $\varphi(a) = \varphi(a)$.

Condition (3.4) arises in the theory of operator algebras in a natural way [4, Definition 1.8].

Remark 3.3.6. A harmonic function φ is semifinite if and only if there exists an element $a \in \mathrm{K}_0^+(\Gamma)$ with $\varphi(a) = +\infty$ and for any such a we can find a sequence $\{a_n\}_{n\geq 1} \subset \mathrm{K}_0^+(\Gamma)$ such that

- $a_n \leq_K a$,
- $\varphi(a_n) < +\infty$,
- $\lim\limits_{n\to+\infty} \varphi(a_n) = +\infty$.

We will call this $\{a_n\}_{n\geq 1}$ *an approximating sequence.*

Proposition 3.3.7. *A harmonic function φ is semifinite if and only if it is not finite and for any vertex $\lambda \in \Gamma$ the following equality holds*

$$\varphi(\lambda) = \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\geq\lambda,\ |\mu|=N \\ 0<\varphi(\mu)<+\infty}} \dim(\lambda,\mu)\varphi(\mu). \tag{3.5}$$

Proof. If equality (3.5) is fulfilled, then φ is semifinite, since prelimit sums give us an approximating sequence. If φ is semifinite and $\varphi(\lambda) < +\infty$ then equality (3.5) is a trivial consequence of Definition 3.3.1. If $\varphi(\lambda) = +\infty$, then we can find an approximating sequence and Observation 3.3.4 yields that the prelimit expression is unbounded in N. We are left to prove that the limit exists. In fact, we show that the prelimit sequence is non-decreasing in N. Let us denote the prelimit expression by ψ_N.

Next, the function

$$\phi(\lambda) = \begin{cases} \varphi(\lambda), \text{ if } 0 < \varphi(\lambda) < +\infty \\ 0 \text{ otherwise} \end{cases}$$

is *subharmonic*:

$$\phi(\lambda) \leq \sum_{\mu:\ \lambda\nearrow\mu} \varkappa(\lambda,\mu)\,\phi(\mu).$$

Then from

$$\psi_N = \sum_{\mu:\ |\mu|=N} \dim(\lambda,\mu)\phi(\mu)$$

and equality (3.3) it follows that $\psi_1 \leq \psi_2 \leq \psi_3 \leq \ldots$. □

Corollary 3.3.8. *If φ is a semifinite harmonic function on a graded graph Γ, then for any vertex $\lambda \in \Gamma$ with $\varphi(\lambda) = +\infty$ there exists a vertex $\mu \geq \lambda$ such that $0 < \varphi(\mu) < +\infty$.*

Remark 3.3.9. Let $\{c_\mu\}_{\mu\in\Gamma}$ be a tuple of non-negative real "numbers" $c_\mu \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for every vertex $\lambda \in \Gamma$ there exists the limit $\lim\limits_{N\to\infty} \sum\limits_{\mu\in\Gamma_N} \dim(\lambda,\mu)c_\mu$, which may be infinite. For instance, we may take $c_\mu = \psi(\mu)$, where ψ is a *subharmonic* function: $\psi(\lambda) \leq \sum\limits_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)\psi(\mu)$.

Then the function

$$\bar{c}(\lambda) = \lim_{N\to\infty} \sum_{\mu\in\Gamma_N} \dim(\lambda,\mu)c_\mu$$

is harmonic, cf. [16, p.4], see also [9, formula (47)].

Definition 3.3.10. A semifinite harmonic function φ is called *indecomposable*, if for any finite or semifinite harmonic function φ' which does not vanish identically on the finiteness ideal of φ and satisfies the inequality $\varphi' \le \varphi$ we have $\varphi' = \text{const}\cdot\varphi$ on the finiteness ideal of φ.

At the first glance the finiteness ideal of φ' might be bigger than that of φ, but the next remark shows that this is not the case.

Remark 3.3.11. If φ and φ' from Definition 3.3.10 are proportional on the finiteness ideal of φ, then they are proportional on the whole graph Γ. Indeed, by virtue of Proposition 3.3.7 we may write

$$\varphi(\lambda) = \text{const}^{-1}\cdot \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\ge\lambda,\ |\mu|=N\\ 0<\varphi(\mu)<+\infty}} \dim(\lambda,\mu)\varphi'(\mu) \le$$

$$\le \text{const}^{-1}\cdot \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\ge\lambda,\ |\mu|=N\\ 0<\varphi'(\mu)<+\infty}} \dim(\lambda,\mu)\varphi'(\mu) = \text{const}^{-1}\cdot\varphi'(\lambda).$$

Thus, $\varphi' \le \varphi \le \text{const}^{-1}\cdot\varphi'$ and finitiness ideals of φ and φ' coincide.

Notation. The set of all indecomposable finite (not identically zero) and semifinite harmonic functions on a graded graph Γ is denoted by $\mathcal{H}_{\text{ex}}(\Gamma)$. The subset of $\mathcal{H}_{\text{ex}}(\Gamma)$ consisting of strictly positive functions is denoted by $\mathcal{H}^{\circ}_{\text{ex}}(\Gamma)$.

Lemma 3.3.12. *Let I be an ideal of a graded graph Γ. Assume that $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)$ does not vanish on I identically. Then the following equality holds*

$$\varphi(\lambda) = \lim_{N\to\infty} \sum_{\substack{\mu\in I\\ |\mu|=N}} \dim(\lambda,\mu)\varphi(\mu), \quad \lambda\in\Gamma. \tag{3.6}$$

Moreover, for any element $a \in \mathrm{K}_0^+(\Gamma)$ we have $\varphi(a) = \sup\limits_{\substack{b\in \mathrm{K}_0^+(I):\ b\le_K a,\\ \varphi(b)<+\infty}} \varphi(b).$

Remark 3.3.13. If we omit the assumption that φ is indecomposable, then the equality above should be replaced by the inequality

$$\varphi(\lambda) \ge \lim_{N\to\infty} \sum_{\substack{\mu\in I\\ |\mu|=N}} \dim(\lambda,\mu)\varphi(\mu).$$

Proof of Lemma 3.3.12. First of all, we remark that there exists a vertex $\nu \in I$ such that $0 < \varphi(\nu) < +\infty$. Indeed, φ does not equal zero identically on I, hence we can find a vertex $\nu' \in I$ such that $\varphi(\nu') > 0$. If $\varphi(\nu') = +\infty$, then by Corollary 3.3.8 we can find another vertex $\nu > \nu'$ with $0 < \varphi(\nu) < +\infty$, which necessarily lies in I.

Note that the function

$$\phi(\lambda) = \begin{cases} \varphi(\lambda), \text{ if } \lambda\in I,\\ 0 \text{ otherwise}\end{cases}$$

is subharmonic on Γ. Then by Remark 3.3.9 the right-hand side of (3.6) defines a harmonic function on Γ. From Observation 3.3.4 and Remark 3.3.6 it follows that the restriction of φ to the ideal I is a finite or semifinite harmonic function on I. Then the harmonic function on Γ defined by the right-hand side of (3.6) is finite or semifinite as well. Next, by the very definition of harmonic functions, the prelimit expression is majorized by φ for any N. Then the harmonic function that is defined as the limit $N \to +\infty$ is also majorized by φ. Finally, indecomposibility of φ implies that φ and the right-hand side of (3.6) are proportional, but they coincide on the ideal I. Thus, they coincide on the whole graph Γ, since there exists $\nu \in I$ with $0 < \varphi(\nu) < +\infty$. □

Now we are ready to prove the most crucial statement of Wassermann's method. The following theorem is a combinatorial analog of a result, which is well known in the context of C^*-algebras, see [31, Theorem 7 p.143, Corollary p.144] and [1, II.6.1.6 p.102].

Theorem 3.3.14. *Let I be an ideal of a graded graph Γ.*

1) There is a bijective correspondence between $\{\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)\colon \varphi|_I \neq 0\}$ and $\mathcal{H}_{\text{ex}}(I)$, defined by the following mutually inverse maps

$$\begin{aligned} &\mathrm{Res}_I^\Gamma : \{\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)\colon \varphi|_I \neq 0\} \to \mathcal{H}_{\text{ex}}(I),\ \ \varphi \mapsto \varphi\Big|_I, \\ &\mathrm{Ext}_I^\Gamma : \mathcal{H}_{\text{ex}}(I) \to \{\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)\colon \varphi|_I \neq 0\},\ \ \varphi(\cdot) \mapsto \lim_{N\to\infty} \sum_{\substack{\mu\in I \\ |\mu|=N}} \dim(\cdot,\mu)\varphi(\mu). \end{aligned}$$

Furthermore, for any element $a \in \mathrm{K}_0^+(\Gamma)$ we have $\mathrm{Ext}_I^\Gamma(\varphi)(a) = \sup\limits_{\substack{b\in \mathrm{K}_0^+(I)\colon b\le_K a, \\ \varphi(b)<+\infty}} \varphi(b).$

2) If Γ is a primitive graded graph, then the bijection above preserves strictly positive harmonic functions $\mathcal{H}^\circ_{\text{ex}}(I) \longleftrightarrow \mathcal{H}^\circ_{\text{ex}}(\Gamma)$.

Proof. Suppose that $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma)$ and $\varphi|_I \neq 0$. Then from Observation 3.3.4 and Remark 3.3.6 it follows that $\mathrm{Res}_I^\Gamma(\varphi) = \varphi|_I$ is a finite or semifinite harmonic function on I. Lemma 3.3.12 implies that $\mathrm{Res}_I^\Gamma(\varphi)$ is indecomposable.

Now let $\varphi \in \mathcal{H}_{\text{ex}}(I)$. From the proof of Proposition 3.3.7 it follows that the limit from the definition of Ext_I^Γ exists and $\mathrm{Ext}_I^\Gamma(\varphi)$ is a finite or semifinite harmonic function on Γ. Note that $\mathrm{Ext}_I^\Gamma(\varphi)$ is strictly positive for $\varphi \in \mathcal{H}^\circ_{\text{ex}}(I)$ because of the following simple fact, which holds for any primitive graded graph. For any vertex $\lambda \in \Gamma$ there exists a vertex $\mu \in I$ such that $\mu \ge \lambda$.

Let us show that the harmonic function $\mathrm{Ext}_I^\Gamma(\varphi)$ is indecomposable for any $\varphi \in \mathcal{H}_{\text{ex}}(I)$. Suppose that $\mathrm{Ext}_I^\Gamma(\varphi) \ge \psi$, for some ψ, that does not vanish on the finiteness ideal of $\mathrm{Ext}_I^\Gamma(\varphi)$ identically. We denote that ideal by $\widetilde{I}$. The finiteness ideal of φ is denoted by I^φ. Let us introduce more notation: $\psi_1 = \psi|_{\widetilde{I}}$ and $\psi_2 = \mathrm{Ext}_I^{\widetilde{I}}(\varphi) - \psi_1$. Then ψ_1 and ψ_2 are finite harmonic functions on $\widetilde{I}$. Note that $\mathrm{Ext}_I^{\widetilde{I}}(\varphi) = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\varphi)$. On the one hand, we have $\mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\varphi) = \psi_1 + \psi_2$. On the other hand, $\varphi = \psi_1 + \psi_2$ on $I \cap I^\varphi$, hence

$$\mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\varphi) = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi_1) + \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi_2) \le \psi_1 + \psi_2,$$

where the last inequality follows from Remark 3.3.13. Therefore, $\psi_1 = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi_1)$ and $\psi_2 = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi_2)$. Let us rewrite the first equality in the form $\psi|_{\widetilde{I}} = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi)$. Then we see that the function $\psi|_{I^\varphi}$ is not equal to zero identically. Now indecomposability of φ yields that φ and ψ are proportional on I^φ. Thus, from $\psi|_{\widetilde{I}} = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\psi)$ and $\mathrm{Ext}_I^{\widetilde{I}}(\varphi) = \mathrm{Ext}_{I\cap I^\varphi}^{\widetilde{I}}(\varphi)$ it follows that $\mathrm{Ext}_I^\Gamma(\varphi)$ and ψ are proportional on $\widetilde{I}$.

Therefore, maps Res_I^Γ and Ext_I^Γ are well defined and the following identity $\mathrm{Res}_I^\Gamma \circ \mathrm{Ext}_I^\Gamma = \mathrm{id}$ holds. The remaining identity $\mathrm{Ext}_I^\Gamma \circ \mathrm{Res}_I^\Gamma = \mathrm{id}$ immediately follows from Lemma 3.3.12. □

Remark 3.3.15. Let $I_1 \subset I_2$ be ideals of Γ. Then $\mathrm{Ext}_{I_2}^{\Gamma} \circ \mathrm{Ext}_{I_1}^{I_2} = \mathrm{Ext}_{I_1}^{\Gamma}$.

Proposition 3.3.16. *[14, p.35 Lemma 12] Let Γ be a graded graph. If $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$, then the support* $\mathrm{supp}(\varphi)$ *is a primitive coideal.*

Proof. Let $\lambda_1, \lambda_2 \in \mathrm{supp}(\varphi)$. Then Lemma 3.3.12 yields

$$\varphi(\lambda_2) = \lim_{N\to\infty} \sum_{\substack{\mu\in\Gamma^{\lambda_1}\\ |\mu|=N}} \dim(\lambda_2,\mu)\varphi(\mu),$$

where $\Gamma^{\lambda_1} = \{\nu \in \Gamma \mid \nu \geq \lambda_1\}$. Then the inequality $\varphi(\lambda_2) > 0$ implies that there exists a vertex μ such that $\mu \geq \lambda_1, \lambda_2$ and $\varphi(\mu) \neq 0$. Thus, by virtue of Proposition 3.2.6 the coideal $\mathrm{supp}(\varphi)$ is primitive. □

3.4 Multiplicative branching graphs

In this section we recall some basic notions related to multiplicative branching graphs [10, 14]. For such graphs we prove a theorem, which states that some multiplicative branching graphs admit no strictly positive semifinite indecomposable harmonic functions [31, Theorem 8 p.146]. We call this theorem Wassermann's no-go theorem. We also prove a semifinite analog of the Vershik-Kerov ring theorem [15, Theorem p.144].

Definition 3.4.1. [14, p.40] A branching graph Γ is called *multiplicative*, if there exists an associative $\mathbb{Z}_{\geq 0}$-graded $\mathbb{R}$-algebra $A = \bigoplus_{n\geq 0} A_n$, $A_0 = \mathbb{R}$ with a distinguished basis of homogeneous elements $\{a_\lambda\}_{\lambda\in\Gamma}$, that satisfy the following conditions

1) $\deg a_\lambda = |\lambda|$
2) $a_\varnothing$ is the identity in A
3) For $\widehat{a} = \sum_{\nu\in\Gamma_1} \varkappa(\varnothing,\nu)a_\nu$ and any vertex $\lambda \in \Gamma$ we have $\widehat{a} \cdot a_\lambda = \sum_{\mu:\lambda\nearrow\mu} \varkappa(\lambda,\mu)a_\mu$.

Moreover, we assume that the structure constants of A with respect to the basis $\{a_\lambda\}_{\lambda\in\Gamma}$ are non-negative.

Let $(\Gamma,\varkappa)$ be the multiplicative graph that is related to an algebra A and a basis $\{a_\lambda\}_{\lambda\in\Gamma}$. We denote the quotient algebra $A/(\widehat{a}-1)$ by R, the canonical homomorphism $A \twoheadrightarrow R$ by $[\cdot]$ and the positive cone in R, consisting of all elements that can be written in the form $\sum_{\lambda\in\Gamma_n} c_\lambda[a_\lambda]$ for a large enough n and some $c_\lambda \geq 0$, by R^+. The correspondence $[\lambda] \mapsto [a_\lambda]$ defines an isomorphism of $\mathbb{R}$-vector spaces $\mathrm{K}_0(\Gamma) \xrightarrow{\sim} R$. The image of the cone $\mathrm{K}_0^+(\Gamma) \subset \mathrm{K}_0(\Gamma)$ under this map coincides with R^+.

Consider the positive cone $A^+ \subset A$, consisting of all elements of A, that can be written as a linear combination of basis elements a_λ with non-negative coefficients. For any semifinite harmonic function $\varphi \in \mathcal{H}(\Gamma)$ we may speak about the $\mathbb{R}_{\geq 0}$-linear map $\varphi\colon A^+ \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

Let us now formulate the Vershik-Kerov ring theorem [15, Theorem p.134], see also [7, Proposition 8.4].

Definition 3.4.2. A harmonic function φ on a branching graph Γ is called *normalized* if $\varphi(\varnothing) = 1$.

Theorem 3.4.3 (Vershik-Kerov Ring Theorem). *[15, Theorem p.134] A finite normalized harmonic function φ on the multiplicative branching graph Γ is indecomposable if and only if the corresponding functional on A is multiplicative: $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)\ \forall a, b \in A$.*

The following semifinite analog of the ring theorem holds.

Theorem 3.4.4. *[15, Theorem p.144] For any semifinite indecomposable harmonic function φ on the multiplicative branching graph Γ there exists a finite normalized indecomposable harmonic function ψ, such that $\varphi(a \cdot b) = \psi(a) \cdot \varphi(b)$ for any $a, b \in A^+$ with $\varphi(b) < +\infty$.*

Proof. Note that
$$(\widehat{a})^n = \sum_{\nu \in \Gamma_n} \dim(\nu) \cdot a_\nu.$$
Then $\varphi\Big((\widehat{a})^n a_\mu\Big) = \varphi\left(a_\mu\right) \geq \dim(\lambda)\varphi\left(a_\lambda \cdot a_\mu\right)$ and $\varphi^\lambda(\mu) = \varphi\left(a_\lambda a_\mu\right)$ is a finite harmonic function on the finiteness ideal of φ. Since the restriction of φ to its finiteness ideal is an indecomposable harmonic function (see Lemma 3.3.12) it follows that there exists $c_\lambda \in \mathbb{R}_{\geq 0}$ such that $\varphi\left(a_\mu \cdot a_\lambda\right) = c_\lambda \varphi(a_\mu)$. We set $\psi(\lambda) = c_\lambda$ by definition. One can check that ψ is a harmonic function and that the functional on A defined by ψ is multiplicative. Then the Vershik-Kerov ring theorem implies that ψ is indecomposable. □

From Theorem 3.4.4 it follows that the subspace $I = \operatorname{span}_{\mathbb{R}}(a_\lambda \mid \lambda\colon \varphi(\lambda) < +\infty) \subset A$ is an ideal for any semifinite indecomposable harmonic function φ. However, the proof shows that this is true for an arbitrary harmonic function φ without any additional assumptions.

The following theorem imposes some restrictions on multiplicative graphs that possess strictly positive indecomposable semifinite harmonic functions, [31, Theorem 8 p.146].

Theorem 3.4.5 (Wassermann's no-go theorem). *If $a_\lambda a_\mu \neq 0$ for any $\lambda, \mu \in \Gamma$, then the graph Γ admits no strictly positive semifinite indecomposable harmonic functions.*

Proof. Let φ be a strictly positive indecomposable semifinite harmonic function. The argument given at the beginning of the proof of Theorem 3.4.4 shows that φ^μ defined by $\varphi^\mu(\lambda) = \varphi\left(a_\lambda a_\mu\right)$ is a finite harmonic function on Γ, while $\varphi(\mu) < +\infty$. Furthermore, the following inequality holds $\varphi \geq \text{const} \cdot \varphi^\mu$. Next, observe that φ^μ is strictly positive, since $a_\lambda a_\mu \neq 0$ and structure constants of A are non-negative with respect to the basis $\{a_\lambda\}_{\lambda \in \Gamma}$. Therefore, φ and φ^μ are proportional. Thus, φ is finite. □

Corollary 3.4.6. *[3, p. 371, the paragraph just before Theorem 3.5] If Γ admits a strictly positive indecomposable finite harmonic function, then it possesses no strictly positive semifinite indecomposable harmonic functions.*

Proof. Suppose that φ is a strictly positive indecomposable finite harmonic function and $a_\lambda a_\mu = 0$ for some $\lambda, \mu \in \Gamma$. Then $\varphi(a_\lambda a_\mu) = \varphi(0) = 0$ and Theorem 3.4.3 yields $\varphi(\lambda)\varphi(\mu) = 0$, which contradicts the strict positivity of φ. □

3.5 Boyer's Lemma

In this section we discuss a very useful claim related to arbitrary harmonic functions on a graded graph. It allows one to determine the finiteness ideal of an indecomposable semifinite harmonic function in several concrete situations. This principle, which was first observed by R. P. Boyer and published only in 1983, see [4, Theorem 1.10, Example p.212], had been also stated by Wassermann [31, Boyer's Lemma p.149] two years before the paper [4]. We formulate and prove a slightly involved generalization of Wassermann's concise argument. It turns out to be a combinatorial analog of [4, Theorem 1.10]. After that we consider a couple of examples, which immediately follow from the general claim. Boyer's Lemma from [31] becomes a part of the first example, see Remark 3.5.6.

3.5.1 General statement

Recall that the set of vertices lying on the n-th level of a graded graph Γ is denoted by Γ_n. Below we work with arbitrary harmonic functions and do *not* assume that they are finite or semifinite.

Definition 3.5.1. A harmonic function φ is called *semifinite at a vertex λ*, if $\varphi(\lambda) = +\infty$ and there exists a sequence $\{a_n\}_{n\geq 1} \subset \mathrm{K}_0^+(\Gamma)$ such that

- $a_n \leq_K \lambda$,
- $\varphi(a_n) < +\infty$,
- $\lim\limits_{n\to+\infty} \varphi(a_n) = +\infty$.

The sequence $\{a_n\}_{n\geq 1}$ will be called *an approximating sequence for the vertex λ.*

Observation 3.5.2. If φ is semifinite at a vertex λ, then for any vertex $\mu \leq \lambda$ the function φ is semifinite at the vertex μ too.

Proposition 3.5.3 (Boyer's lemma). *Let Γ be a graded graph and φ be a harmonic function on it. Assume that $I \subset \Gamma$ is an ideal, $J = \Gamma \backslash I$ is the corresponding coideal and we are given a fixed vertex $\lambda \in J$. Suppose that there exists a vertex $\lambda' \in I$ and a positive real number β_λ such that $\varphi(\lambda') > 0$ and for any vertex $\eta \in I$ lying on a large enough level the following inequality holds*

$$\sum_{\mu\in J} \dim(\lambda,\mu)\varkappa(\mu,\eta) \geq \beta_\lambda \dim(\lambda',\eta). \tag{3.7}$$

Then $\varphi(\lambda) = +\infty$. If in addition φ is semifinite at λ', then φ is semifinite at λ as well.

Remark 3.5.4. Condition (3.7) is a refinement of some condition on the "number" of paths in the graph Γ, which admits a graphical interpretation, see condition (3.14) from Corollary 3.5.5 and Figure 3.2.

Proof of Proposition 3.5.3. Let us multiply (3.7) by $\eta \in K_0(\Gamma)$ and sum over all $\eta \in I_m$ for some m. Then we get

$$\sum_{\substack{\eta\in I_m \\ \mu\in J}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\cdot\eta \geq_K \sum_{\eta\in I_m} \beta_\lambda \dim(\lambda',\eta)\cdot\eta = \beta_\lambda\cdot\lambda',$$

where the both sides of the inequality are considered as elements of $\mathrm{K}_0(\Gamma)$ and the partial order on $\mathrm{K}_0(\Gamma)$ defined by the cone $\mathrm{K}_0^+(\Gamma)$ is denoted by $\geq_K$.

The only thing we are left to do is to reproduce the original argument of A. Wassermann [31, p.149, the proof of Boyer's Lemma] in our context. Let us set $n = |\lambda|$, then

$$\lambda = \sum_{\overline{\eta}\in\Gamma_{n+N+1}} \dim(\lambda,\overline{\eta})\overline{\eta} \geq_K \sum_{\overline{\eta}\in I_{n+N+1}} \dim(\lambda,\overline{\eta})\overline{\eta} \tag{3.8}$$

for any N.

Note that, if $\lambda \in J_n$ and $\overline{\eta} \in I_{n+N+1}$, then

$$\dim(\lambda,\overline{\eta}) = \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1} \\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\dim(\eta,\overline{\eta}). \tag{3.9}$$

Substitute (3.9) into (3.8): $\lambda \geq_K \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \sum_{\overline{\eta}\in I_{n+N+1}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\dim(\eta,\overline{\eta})\overline{\eta}$.

Now sum over $\overline{\eta}$:

$$\lambda \geq_K \sum_{l=0}^{N} \sum_{\substack{\eta\in I_{n+l+1}\\ \mu\in J_{n+l}}} \dim(\lambda,\mu)\varkappa(\mu,\eta)\eta \geq_K N\cdot\beta_\lambda\lambda' \tag{3.10}$$

Compare (3.10) with (1.10.1) and (1.10.2) from [4, Theorem 1.10].

Thus, (3.10) yields $\varphi(\lambda) \geq \beta_\lambda\varphi(\lambda')\cdot N$ for any N hence $\varphi(\lambda) = +\infty$. Moreover, the sequence $a_N = N\cdot\beta_\lambda\lambda'$ is an approximating sequence for the vertex λ if $\varphi(\lambda') < +\infty$. □

3.5.2 Example 1

Consider graded graphs $(\Gamma_1,\varkappa_1)$ and $(\Gamma_2,\varkappa_2)$ and suppose that we are given a graded map $\Gamma_1 \to \Gamma_2$, $\lambda \mapsto \lambda'$. Let $(\Gamma,\varkappa)$ be one more graded graph that satisfies the following requirements:

$$(\Gamma)_n = (\Gamma_1)_n \sqcup (\Gamma_2)_{n-1} \text{ for } n \geq 1,\ (\Gamma)_0 = (\Gamma_1)_0. \tag{3.11}$$

$$\begin{aligned} \varkappa(\lambda,\mu) &= \varkappa_1(\lambda,\mu), \text{ if } \lambda,\mu \in \Gamma_1,\\ \varkappa(\lambda,\mu) &= \varkappa_2(\lambda,\mu), \text{ if } \lambda,\mu \in \Gamma_2. \end{aligned} \tag{3.12}$$

$$\varkappa(\lambda,\mu) = 0, \text{ if } \lambda \in \Gamma_2, \mu \in \Gamma_1. \tag{3.13}$$

Condition (3.13) means that Γ_2 is an ideal of Γ. For simplicity one can assume that edges from Γ_1 to Γ_2 can go from λ to λ' only, see Figure 3.1. But we will not use this later on.

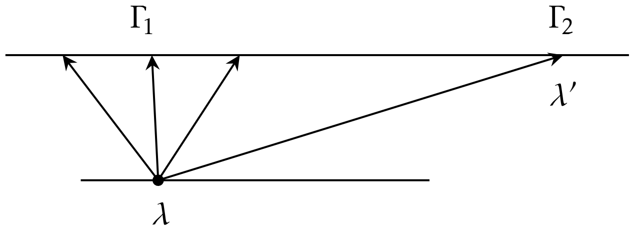


Figure. 3.1: Example of the branching rule for Γ.

Corollary 3.5.5. *Assume that the map $\nu \mapsto \nu'$ is surjective and let $\lambda \in (\Gamma_1)_n$ be a fixed vertex. Suppose that for any large enough l and any vertex $\mu \in (\Gamma_1)_{n+l}$ the following inequality holds*

$$\dim_1(\lambda,\mu)\varkappa(\mu,\mu') \geq \dim_2(\lambda',\mu'), \tag{3.14}$$

where $\dim_1(\cdot,\cdot)$ and $\dim_2(\cdot,\cdot)$ are shifted dimensions for $(\Gamma_1,\varkappa_1)$ and $(\Gamma_2,\varkappa_2)$. Now let φ be a harmonic function on Γ with $\varphi(\lambda') > 0$. Then $\varphi(\lambda) = +\infty$.

Proof. Recall that Γ_2 is an ideal of Γ. Therefore, we may apply Proposition 3.5.3 for $I = \Gamma_2$, $J = \Gamma_1$, and $\beta_\lambda = 1$. For that we bound from below the sum in the left hand side of (3.7) in terms of one of its summands and use (3.14). □

Remark 3.5.6. If the map $\lambda \mapsto \lambda'$ is a branching graph morphism, that is $\varkappa(\lambda,\mu) = \varkappa(\lambda',\mu')$, then condition (3.14) means that $\varkappa(\mu,\mu') \geq 1$. If the equality holds identically, then we obtain the original formulation of Boyer's Lemma [31, p.149, Boyer's lemma].

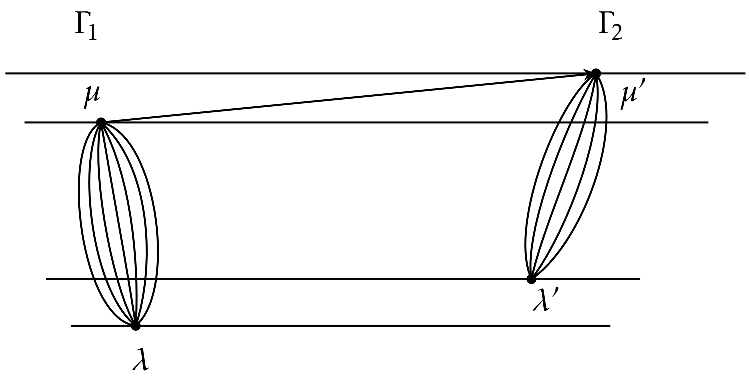


Figure. 3.2: Condition (3.14) means that the "number" of paths from λ to μ', that go through μ, is not smaller then the "number" of arbitrary paths from λ' to μ'.

3.5.3 Example 2

Let us consider graded graphs $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$ and suppose that we are given a graded map $\Gamma_1 \to \Gamma_2$, $\lambda \mapsto \lambda'$. Let $(\Gamma, \varkappa)$ be another graded graph, that satisfies $(\Gamma)_n = (\Gamma_1)_n \sqcup (\Gamma_2)_n$ for $n \geq 0$, and conditions (3.12), (3.13). Recall that the last condition means that Γ_2 is an ideal of Γ. For simplicity one can assume that vertices $\lambda \in \Gamma_1$ and $\mu \in \Gamma_2$ are joined by an edge if and only if $\lambda' \nearrow \mu$, as it is shown on Figure 3.3.

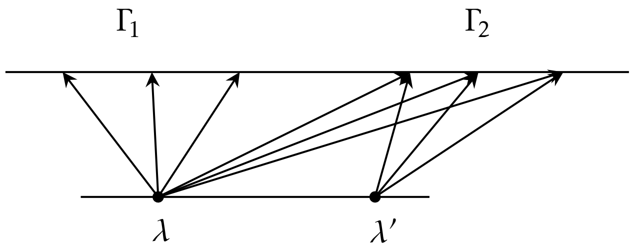


Figure. 3.3: Example of the branching rule for Γ.

Corollary 3.5.7. *Suppose that the map $\lambda \mapsto \lambda'$ is surjective. Let $\lambda \in \Gamma_1$ be a fixed vertex and assume that the following inequalities hold for any $\mu \in \Gamma_1$*

$$\varkappa(\lambda, \mu) \geq \varkappa(\lambda', \mu'),$$
$$\varkappa(\lambda, \mu') \geq \varkappa(\lambda', \mu').$$

Then $\varphi(\lambda) = +\infty$ for any harmonic function φ on Γ such that $\varphi(\lambda') > 0$.

Proof. Let us take $I = \Gamma_2$, $J = \Gamma_1$, and $\beta_\lambda = 1$ in Proposition 3.5.3 and prove that the following inequality holds $\sum_{\mu \in \Gamma_1} \dim(\lambda, \mu)\varkappa(\mu, \eta) \geq \dim(\lambda', \eta)$ for any $\eta \in \Gamma_2$. In order to do so, we check that $\dim(\lambda, \mu) \geq \dim(\lambda', \mu')$ and write

$$\frac{\sum_{\mu \in \Gamma_1} \dim(\lambda, \mu)\varkappa(\mu, \eta)}{\dim(\lambda', \eta)} \geq \frac{\sum_{\mu \in \Gamma_1} \dim(\lambda', \mu')\varkappa(\mu', \eta)}{\dim(\lambda', \eta)} \geq \frac{\sum_{\overline{\mu} \in \Gamma_2} \dim(\lambda', \overline{\mu})\varkappa(\overline{\mu}, \eta)}{\dim(\lambda', \eta)} = 1.$$

For each of these inequalities we have used that $\lambda \mapsto \lambda'$ is surjective. □

Remark 3.5.8. One can obtain an exhaustive list of indecomposable semifinite harmonic functions on the Macdonald graph, which corresponds to the simplest Pieri rule for the Macdonald symmetric functions, by applying Wassermann's method. This list turns out to be very similar to that for the Young graph, see [31, Theorem 9 on page 150]. For instance, the space of classification parameters is an obvious (q,t)-deformation of the parameter space for the Young graph. Namely, we should deform only the continuous part of the data in the same way as it is deformed in the case of finite harmonic functions, replacing the ordinary Thoma simplex with the (q,t)-deformed Thoma simplex, see Theorem 1.4 and Proposition 1.6 from [18], while the discrete part remains the same. This result easily follows from the original argument of A. Wassermann, Theorem 1.4 and Proposition 1.6 from [18], Proposition 3.2.6, Theorem 3.3.14, Proposition 3.3.16, and Corollary 3.5.5. Instead of using Theorem 3.4.5 we must apply a similar argument obtained with the help of a trick due to K. Matveev [18, §6, Proof of Proposition 1.6].

Chapter 4

Direct product of branching graphs

4.1 Summary in French

Dans cette section, nous décrivons les fonctions harmoniques finies indécomposables sur le produit de graphes de branchement en termes de fonctions harmoniques sur les multiplicateurs. On peut considérer le résultat principal de cette sous-section, Proposition 4.2.6, comme une généralisation du théorème de de Finetti bien connu [2, Théorème 5.1, Théorème 5.2]. La différence entre la Proposition 4.2.6 (cas $n = 2$) et le théorème de de Finetti est que nous remplaçons les deux côtés du triangle de Pascal, qui correspondent à deux plongements $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ le long des première et deuxième composantes, par des graphes de branchement arbitraires. Remarquons que le cas où l'un de ces graphes est une ligne composée d'un seul sommet à chaque niveau est déjà connu, voir [30, Théorème 2.8]. Notez que dans ce théorème, on ne considère que des fonctions harmoniques strictement positives (ou, de manière équivalente, des mesures centrales) au lieu de fonctions arbitraires.

Donnons maintenant une motivation pour la définition principale de la présente section. Si A et B sont des $\mathbb{Z}_{\geq 0}$-algèbres graduées unitales sur $\mathbb{R}$, alors leur produit tensoriel (sur $\mathbb{R}$) est également une algèbre graduée unitale. Plus précisément, si $A = \bigoplus_{n\geq 0} A_n$, $A_0 = \mathbb{R}$ et $B = \bigoplus_{n\geq 0} B_n$, $B_0 = \mathbb{R}$, alors $A \otimes_{\mathbb{R}} B = \bigoplus_{k\geq 0} (A \otimes_{\mathbb{R}} B)k$, où

$$(A \otimes_{\mathbb{R}} B)_k = \bigoplus_{\substack{n,m\geq 0:\\ n+m=k}} A_n \otimes_{\mathbb{R}} B_m.$$

De plus, $\mathbb{1}_{A\otimes B} = \mathbb{1}_A \otimes \mathbb{1}_B$ et $(A \otimes_{\mathbb{R}} B)_0 = \mathbb{R} \cdot \mathbb{1}_{A\otimes B}$. Ce fait simple, ainsi que la Définition 3.4.1, nous motivent à considérer le *produit direct* de deux graphes gradués.

Ici, nous décrivons les fonctions harmoniques finies et semi-finies sur le produit direct de graphes de branchement en termes de telles fonctions sur les facteurs.

Définition 4.1.1. Par *produit direct* de graphes gradués $(\Gamma_1, \varkappa_1)$ et $(\Gamma_2, \varkappa_2)$, nous entendons le graphe gradué $(\Gamma_1 \times \Gamma_2, \varkappa_1 \times \varkappa_2)$, où

$$(\Gamma_1 \times \Gamma_2)_k = \bigsqcup_{\substack{n,m\geq 0:\\ n+m=k}} (\Gamma_1)_n \times (\Gamma_2)_m$$

et

$$(\varkappa_1 \times \varkappa_2)\Big((\lambda_1, \mu_1); (\lambda_2, \mu_2)\Big) = \begin{cases} \varkappa_1(\lambda_1, \lambda_2), & \text{si } \mu_1 = \mu_2, \\ \varkappa_2(\mu_1, \mu_2), & \text{si } \lambda_1 = \lambda_2, \\ 0 & \text{sinon.} \end{cases}$$

Exemple 4.1.2. Le triangle de Pascal est le produit direct de deux copies de $\mathbb{Z}_{\geq 0}$.

Nous notons par $\mathcal{FH}_{\text{ex}}(\Gamma)$ l'ensemble de toutes les fonctions harmoniques finies normalisées indécomposables sur un graphe de branchement Γ.

Théorème 4.1.3. Soient Γ_1 et Γ_2 des graphes de branchement et φ une fonction harmonique finie normalisée indécomposable sur $\Gamma_1 \times \Gamma_2$, c'est-à-dire $\varphi \in \mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$. Alors un seul des cas suivants peut se produire:

1) Il existe $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$ et des nombres réels positifs w_1, w_2 tels que $w_1 + w_2 = 1$ et

$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda) \varphi_2(\mu). \tag{4.1}$$

De plus, ces φ_1, φ_2, w_1, w_2 sont uniques.

2) Il existe $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$ telle que

$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \mu \neq \varnothing, \\ \varphi_1(\lambda), & \text{si } \mu = \varnothing. \end{cases} \tag{4.2}$$

3) Il existe $\varphi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$ telle que

$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \lambda \neq \varnothing, \\ \varphi_2(\mu), & \text{si } \lambda = \varnothing. \end{cases} \tag{4.3}$$

De plus, chaque fonction harmonique sur $\Gamma_1 \times \Gamma_2$ de la forme 1), 2) ou 3) est indécomposable.

Remarque 4.1.4. On peut facilement voir que (4.2) et (4.3) sont des cas particuliers de (4.1) correspondant à $w_2 = 0$ et $w_1 = 0$. Nous formulons le Théorème 4.1.3 sous cette forme pour simplifier la comparaison avec le Théorème 4.1.5.

Théorème 4.1.5. Soient Γ_1 et Γ_2 des graphes gradués et $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$, alors un seul des cas suivants peut se produire :

1) Il existe $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ et des nombres réels positifs w_1, w_2 avec $w_1 + w_2 = 1$ tels que

$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda) \varphi_2(\mu).$$

De plus, ces φ_1 et φ_2 sont définis de manière unique à une constante multiplicative près.

2) Il existe $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$ et $\nu_2 \in \Gamma_2$ tels que

$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \mu \not\leq \nu_2, \\ +\infty, & \text{si } \mu < \nu_2, \\ \varphi_1(\lambda), & \text{si } \mu = \nu_2. \end{cases}$$

3) Il existe $\nu_1 \in \Gamma_1$ et $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ tels que

$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{si } \lambda \not\leq \nu_1, \\ +\infty, & \text{si } \lambda < \nu_1, \\ \varphi_2(\mu), & \text{si } \lambda = \nu_1. \end{cases}$$

De plus, chaque fonction harmonique sur $\Gamma_1 \times \Gamma_2$ de la forme 1), 2) ou 3) est finie ou semi-finie et indécomposable.

4.2 Finite harmonic functions

In this section we describe indecomposable finite harmonic functions on the product of branching graphs in terms of harmonic functions on the multipliers. One can treat the main result of this subsection, Proposition 4.2.6, as a generalization of the well known de Finetti theorem [2, Theorem 5.1, Theorem 5.2]. The difference between Proposition 4.2.6 ($n = 2$ case) and the de Finetti theorem is that we replace two sides of the Pascal triangle, which correspond to two embeddings $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ along the first and the second components, with arbitrary branching graphs. Remark that the case when one of these graphs is a line consisting of one vertex at each level has been already known, see [30, Theorem 2.8]. Note that in this theorem one should consider only strictly positive harmonic functions (or, equivalently, central measures) instead of arbitrary ones.

Let us provide some motivation for the main definition of the present section. If A and B are unital $\mathbb{Z}_{\geq 0}$-graded $\mathbb{R}$-algebras, then their tensor product (over $\mathbb{R}$) is a unital graded algebra too. Namely, if $A = \bigoplus_{n\geq 0} A_n$, $A_0 = \mathbb{R}$ and $B = \bigoplus_{n\geq 0} B_n$, $B_0 = \mathbb{R}$, then $A \otimes_{\mathbb{R}} B = \bigoplus_{k\geq 0} (A \otimes_{\mathbb{R}} B)_k$, where

$$(A \otimes_{\mathbb{R}} B)_k = \bigoplus_{\substack{n,m\geq 0:\\ n+m=k}} A_n \otimes_{\mathbb{R}} B_m.$$

Furthermore, $\mathbb{1}_{A\otimes B} = \mathbb{1}_A \otimes \mathbb{1}_B$ and $(A \otimes_{\mathbb{R}} B)_0 = \mathbb{R} \cdot \mathbb{1}_{A\otimes B}$. This simple fact, together with Definition 3.4.1, motivates us to consider the *direct product* of two graded graphs.

Definition 4.2.1. By the *direct product* of graded graphs $(\Gamma_1, \varkappa_1)$ and $(\Gamma_2, \varkappa_2)$ we mean the graded graph $(\Gamma_1 \times \Gamma_2, \varkappa_1 \times \varkappa_2)$, where

$$(\Gamma_1 \times \Gamma_2)_k = \bigsqcup_{\substack{n,m\geq 0:\\ n+m=k}} (\Gamma_1)_n \times (\Gamma_2)_m$$

and

$$(\varkappa_1 \times \varkappa_2)\Big((\lambda_1, \mu_1); (\lambda_2, \mu_2)\Big) = \begin{cases} \varkappa_1(\lambda_1, \lambda_2), & \text{if } \mu_1 = \mu_2, \\ \varkappa_2(\mu_1, \mu_2), & \text{if } \lambda_1 = \lambda_2, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma ties together some properties of the direct product of graded graphs.

The subset $\Gamma_\lambda = \{\mu \in \Gamma \mid \mu \leq \lambda\}$ of a graded graph Γ is called *the principle coideal* associated to $\lambda \in \Gamma$.

Lemma 4.2.2. *Let Γ_1 and Γ_2 be graded graphs.*

1) *A graph $\Gamma_1 \times \Gamma_2$ is primitive if and only if Γ_1 and Γ_2 are primitive.*

2) *Let $\lambda, \lambda' \in \Gamma_1$ and $\mu, \mu' \in \Gamma_2$. Then*

$$\dim\Big((\lambda, \mu), (\lambda', \mu')\Big) = \binom{|\lambda'| - |\lambda| + |\mu'| - |\mu|}{|\lambda'| - |\lambda|} \dim_1(\lambda, \lambda') \dim_2(\mu, \mu'),$$

where $\binom{n}{k}$ denotes the binomial coefficient and $\dim_1(\cdot, \cdot)$, $\dim_2(\cdot, \cdot)$ are shifted dimensions for Γ_1 and Γ_2, see (3.2) on page 27.

Proof. The first assertion follows from Proposition 3.2.6 immediately and the second one is obvious. ☐

Lemma 4.2.3. *Let Γ_1 and Γ_2 be graded graphs. If $J \subset \Gamma_1 \times \Gamma_2$ is a saturated primitive coideal, then it is of the form $J = J_1 \times J_2$ for some coideals $J_1 \subset \Gamma_1$ and $J_2 \subset \Gamma_2$ such that*

- *J_1, J_2 are saturated and primitive or*
- *J_1 is principle[1] and J_2 is saturated and primitive or*
- *J_1 is saturated and primitive and J_2 is principle.*

Proof. We can form J_1 and J_2 via natural projections $\Gamma_1 \times \Gamma_2 \to \Gamma_1$ and $\Gamma_1 \times \Gamma_2 \to \Gamma_2$. Then clearly $J \subset J_1 \times J_2$, but primitivity of J implies that $J_1 \times J_2 \subset J$. Thus, $J = J_1 \times J_2$. It is easy to check that J_1 and J_2 are coideals satisfying the following condition: if $\lambda, \mu \in J_1$ (or $\in J_2$), then there exists $\nu \in J_1$ (or $\in J_2$) such that $\nu \geq \lambda, \mu$. By Proposition 3.2.6 it remains to show that J_1 (and similarly J_2) is either saturated or principal. Let us check that if J_1 is not saturated then it is principal. If J_1 is not saturated then there exists a vertex $\lambda \in J_1$ such that none of the vertices lying above λ belongs to J_1. Let μ be an arbitrary vertex $\mu \in J_1$. Then the property above states that there exists a vertex $\nu \in J_1$ such that $\nu \geq \lambda, \mu$. But this implies that $\nu = \lambda$. Thus, J_1 is principal. □

Remark 4.2.4. Note that the direct product of multiplicative graphs is multiplicative too. For the direct product of two multiplicative graphs the corresponding algebra is the tensor product of the initial algebras, the distinguished basis is the tensor product of the bases and the element that was denoted by $\widehat{a}$ in Definition 3.4.1 is $\widehat{a} \otimes_{\mathbb{R}} \mathbb{1}_B + \mathbb{1}_A \otimes_{\mathbb{R}} \widehat{b}$, where $\widehat{a}$ and $\widehat{b}$ are the same elements for the initial algebras. Thus, we can define the direct product of finitely many graded graphs and the product of multiplicative graphs is multiplicative as well.

Recall that a harmonic function φ on a branching graph Γ is called normalized if $\varphi(\varnothing) = 1$.

Remark 4.2.5. Let $\Gamma_1, \dots, \Gamma_n$ be branching graphs and let $\varphi_1, \dots, \varphi_n$ be finite normalized harmonic functions on them. Then the function $\varphi\colon \Gamma_1 \times \ldots \times \Gamma_n \to \mathbb{R}_{\geq 0}$ defined by

$$\varphi(\lambda_1, \dots, \lambda_n) = w_1^{|\lambda_1|} \dots w_n^{|\lambda_n|} \varphi_1(\lambda_1) \dots \varphi_n(\lambda_n) \tag{4.4}$$

is harmonic and normalized whenever $w_1, \dots, w_n \in \mathbb{R}_{\geq 0}$ and $w_1 + \ldots + w_n = 1$.

Notation. Let $(\Gamma, \varkappa)$ be a branching graph. We denote by $\mathcal{FH}_{\mathrm{ex}}(\Gamma)$ the set of all finite normalized indecomposable harmonic functions on Γ and by $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma)$ the subset of all strictly positive functions.

Proposition 4.2.6. *Let $\Gamma_1, \dots, \Gamma_n$ be branching graphs and Δ_n^0 be the interior of the $n-1$-dimensional simplex, i.e. $\Delta_n^0 = \{(w_1, \dots, w_n) \mid w_1 + \ldots + w_n = 1,\ w_i > 0\}$.*

1) There is a bijection between $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ and $\mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\mathrm{ex}}(\Gamma_n) \times \Delta_n^0$ defined by (4.4).

2) There is a bijection between $\mathcal{FH}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ and $\displaystyle\bigsqcup_{\substack{I\,:\, I \subset \{1,2,\dots,n\} \\ I \neq \varnothing}} \Delta^0_{|I|} \times \bigtimes_{i \in I} \mathcal{FH}_{\mathrm{ex}}(\Gamma_i)$.

More precisely, for any harmonic function $\varphi \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_1 \times \ldots \times \Gamma_n)$ there exist a non-empty set $I \subset \{1, 2, \dots, n\}$, harmonic functions $\varphi_i \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_i)$, which are indexed by $i \in I$, and $w \in \Delta^0_{|I|}$ such that for any n-tuple of vertices $\lambda_1 \in \Gamma_1, \dots, \lambda_n \in \Gamma_n$ the following identity holds

$$\varphi(\lambda_1, \dots, \lambda_n) = \begin{cases} \prod\limits_{i \in I} w_i^{|\lambda_i|} \varphi_i(\lambda_i), & \text{if } \lambda_j = \varnothing,\ \forall j \in \{1, 2, \dots, n\} \setminus I, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, these I, φ_i and w are uniquely defined.

[1] We say that a coideal of a graded graph Γ is principal if it is of the form $\{\lambda \in \Gamma \mid \lambda \leq \mu\}$ for some $\mu \in \Gamma$.

Example 4.2.7. Let us take $\Gamma_1 = \ldots = \Gamma_n = \mathbb{Z}_{\geq 0}$ and assume that all edges are simple and go from k to $k+1$ for $k \geq 0$. Then $\mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_i) = \mathcal{FH}_{\text{ex}}(\Gamma_i)$ is a singleton and $\Gamma_1 \times \ldots \times \Gamma_n$ is the Pascal pyramid $\mathbb{P}_n$. Then from Proposition 4.2.6 it follows that $\mathcal{FH}^{\circ}_{\text{ex}}(\mathbb{P}_n) = \Delta^0_n$ and $\mathcal{FH}_{\text{ex}}(\mathbb{P}_n) = \bigsqcup\limits_{\substack{I\colon I\subset\{1,2,\ldots,n\}\\ I\neq\varnothing}} \Delta^0_{|I|} = \Delta_n$, which is the $n-1$-dimensional simplex.

Remark 4.2.8. For multiplicative graphs Proposition 4.2.6 is a straightforward consequence of the Vershik-Kerov ring theorem (Theorem 3.4.3). Namely, we should apply this theorem to the following simple fact

$$\mathrm{Hom}\Big(A_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} A_n, \mathbb{R}\Big) = \bigtimes_{i=1}^{n} \mathrm{Hom}\,(A_i, \mathbb{R}),$$

where Hom stands for the set of algebra homomorphisms. Indeed, to prove the first part of the proposition we note that there are two mutually inverse maps

$$\Phi_{\rightarrow}\colon\ \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_n) \times \Delta^0_n \longrightarrow \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1 \times \ldots \times \Gamma_n),$$
$$(\varphi_1, \ldots, \varphi_n, w) \mapsto \big(\varphi_1 \circ r_{w_1}\big) \otimes \ldots \otimes \big(\varphi_n \circ r_{w_n}\big);$$

and

$$\Phi_{\leftarrow}\colon\ \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1 \times \ldots \times \Gamma_n) \longrightarrow \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1) \times \ldots \times \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_n) \times \Delta^0_n,$$
$$\varphi \mapsto \big(\varphi\,|_{A_1} \circ r^{-1}_{w_1}, \ldots, \varphi\,|_{A_n} \circ r^{-1}_{w_n}, w\big).$$

Here r_u denotes the automorphism of a graded algebra defined on homogeneous elements as $a \mapsto u^{\deg a} a$, and $\varphi\,|_{A_i}$ is the restriction of $\varphi\colon A_1 \otimes \ldots \otimes A_n \to \mathbb{R}$ to the subalgebra $1^{\otimes i-1} \otimes A_i \otimes 1^{\otimes n-i} \simeq A_i$. Furthermore, the n-tuple $w = (w_1, \ldots, w_n)$ that appears in the definition of the map $\Phi_{\leftarrow}$ has the following form $w_i = \varphi\big(1^{\otimes i-1} \otimes \widehat{a}^{(i)} \otimes 1^{\otimes n-i}\big)$. Recall that the element $\widehat{a}^{(i)} \in A_i$ defines the branching rule for Γ_i, see Definition 3.4.1.

Remark 4.2.9. Proposition 4.2.6 gives us the following view on Kerov's construction [7, §4]. Comultiplication provides us a linear map $\mathrm{K}_0(\Gamma) \to \mathrm{K}_0(\underbrace{\Gamma \times \ldots \times \Gamma}_{n})$ and we take the composite of this map with an indecomposable harmonic function on $\underbrace{\Gamma \times \ldots \times \Gamma}_{n}$ to obtain an indecomposable harmonic function on Γ.

We prove the first part of Proposition 4.2.6 for $n = 2$ only. The case $n > 2$ can be dealt with in the same manner.

In the case $n = 2$ the proposition can be restated in a more friendly from.

Theorem 4.2.10. *Let Γ_1 and Γ_2 be branching graphs and φ be a normalized indecomposable finite harmonic function on $\Gamma_1 \times \Gamma_2$, i.e. $\varphi \in \mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$. Then only one of the following situations can occur:*

1) *There exist $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$ and real positive numbers w_1, w_2 with $w_1 + w_2 = 1$ such that*
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda) \varphi_2(\mu). \tag{4.5}$$
Moreover, these φ_1, φ_2, w_1, w_2 are uniquely defined.

2) *There exist $\varphi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \mu \neq \varnothing, \\ \varphi_1(\lambda), & \text{if } \mu = \varnothing. \end{cases} \tag{4.6}$$

3) There exist $\varphi_2 \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_2)$ such that

$$\varphi(\lambda,\mu) = \begin{cases} 0, & \text{if } \lambda \neq \varnothing, \\ \varphi_2(\mu), & \text{if } \lambda = \varnothing. \end{cases} \tag{4.7}$$

Furthermore, every harmonic function on $\Gamma_1 \times \Gamma_2$ of the form 1), 2), or 3) is indecomposable.

Remark 4.2.11. One can readily see that (4.6) and (4.7) are partial cases of (4.5) corresponding to $w_2 = 0$ and $w_1 = 0$. We formulate Theorem 4.2.10 in this form to simplify the comparison with Theorem 4.3.1.

Theorem 4.2.10 immediately follows from the two lemmas below.

Lemma 4.2.12. *Let Γ_1 and Γ_2 be branching graphs and φ be an indecomposable finite normalized harmonic function on their product, i.e. $\varphi \in \mathcal{FH}_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$. Then φ is either of the form 1), or 2), or 3) from Theorem 4.2.10.*

Proof. Recall that by $\mathcal{T}(\Gamma)$ we denote the space of infinite paths in a branching graph Γ starting at $\varnothing$. By [10, p.60, Theorem] there exists a path $\tau = \Big((\varnothing,\varnothing),(\lambda'_1,\mu'_1),(\lambda'_2,\mu'_2),\dots\Big) \in \mathcal{T}(\Gamma_1 \times \Gamma_2)$ such that for any $\lambda \in \Gamma_1$, $\mu \in \Gamma_2$

$$\varphi(\lambda,\mu) = \lim_{N\to+\infty} \frac{\dim\Big((\lambda,\mu),(\lambda'_N,\mu'_N)\Big)}{\dim\Big((\lambda'_N,\mu'_N)\Big)}. \tag{4.8}$$

Next, we can write

$$\frac{\dim\Big((\lambda,\mu),(\lambda'_N,\mu'_N)\Big)}{\dim\Big((\lambda'_N,\mu'_N)\Big)} = \frac{\Big(|\lambda'_N|\Big)^{\downarrow|\lambda|} \cdot \Big(|\mu'_N|\Big)^{\downarrow|\mu|}}{\Big(|\lambda'_N|+|\mu'_N|\Big)^{\downarrow(|\lambda|+|\mu|)}} \cdot \frac{\dim_1\left(\lambda,\lambda'_N\right)}{\dim_1\left(\lambda'_N\right)} \cdot \frac{\dim_2\left(\mu,\mu'_N\right)}{\dim_2\left(\mu'_N\right)},$$

where $x^{\downarrow k} = x(x-1)\dots(x-k+1)$.

Passing to appropriate subsequences of vertices in τ we may assume that the following limits exist

$$\lim_{N\to+\infty} \frac{\dim_1\left(\lambda,\lambda'_N\right)}{\dim_1\left(\lambda'_N\right)}, \lim_{N\to+\infty} \frac{\dim_2\left(\mu,\mu'_N\right)}{\dim_2\left(\mu'_N\right)}, \lim_{N\to+\infty} \frac{|\lambda'_N|}{|\lambda'_N|+|\mu'_N|}, \lim_{N\to+\infty} \frac{|\mu'_N|}{|\lambda'_N|+|\mu'_N|}.$$

Denote them by $\varphi_1(\lambda)$, $\varphi_2(\mu)$, w_1, and w_2. Suppose that w_1 and w_2 are non-zero, this situation corresponds to the case 1) in Theorem 4.2.10. Then it is easy to check that φ_1 and φ_2 are harmonic functions on Γ_1 and Γ_2. They are indecomposable, since φ is indecomposable.

Suppose now that $w_1 = 0$. Then $w_2 = 1$ and one can check that φ_2 is a harmonic function on Γ_2 as before. Equation (4.8) turns into (4.7) from the case 3) in Theorem 4.2.10. By the same argument as before φ_2 is indecomposable. The case $w_2 = 0$ can be dealt with similarly; it corresponds to the case 2) in Theorem 4.2.10. □

It is clear that the functions defined in cases 2) and 3) in Theorem 4.2.10 are indecomposable, if so are φ_1 and φ_2. Below we prove that the function defined in 1) is indecomposable as well, if φ_1 and φ_2 are indecomposable.

For any branching graph Γ we endow $\mathcal{FH}_{\mathrm{ex}}(\Gamma)$ with the pointwise convergence topology, in which it is a metrizable space, since Γ is countable.

Lemma 4.2.13. *Let Γ_1 and Γ_2 be branching graphs, φ_1 and φ_2 be some finite normalized harmonic functions on them, and w_1 and w_2 be some positive real numbers subject to $w_1 + w_2 = 1$. Then the harmonic function φ on the graph $\Gamma_1 \times \Gamma_2$ defined by (4.5) is indecomposable if and only if φ_1 and φ_2 are indecomposable.*

Proof. If φ is indecomposable, then functions φ_1 and φ_2 can not be decomposable by the very definition of φ. Suppose that φ_1 and φ_2 are indecomposable. By Choquet's theorem, see [22] or [21, Theorem 9.2], there exists a unique probability measure P on the set $\mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$, representing φ in the following sence. For any $\lambda_1 \in \Gamma_1$ and $\lambda_2 \in \Gamma_2$ we have

$$\varphi(\lambda_1, \lambda_2) = \int\limits_{\mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)} \psi(\lambda_1, \lambda_2) P(d\psi). \tag{4.9}$$

From Lemma 4.2.12 it follows that indecomposable harmonic functions on $\Gamma_1 \times \Gamma_2$ are of the following three kinds

1) $\psi(\lambda_1, \lambda_2) = u_1^{|\lambda_1|} u_2^{|\lambda_2|} \psi_1(\lambda_1)\psi_2(\lambda_2)$ for some real positive numbers u_1, u_2 with $u_1 + u_2 = 1$ and some $\psi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$, $\psi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$.

2) $\psi(\lambda_1, \lambda_2) = \begin{cases} \psi_1(\lambda_1), & \text{if } \lambda_2 = \varnothing, \\ 0 & \text{otherwise} \end{cases}$ for some $\psi_1 \in \mathcal{FH}_{\text{ex}}(\Gamma_1)$.

3) $\psi(\lambda_1, \lambda_2) = \begin{cases} \psi_2(\lambda_2), & \text{if } \lambda_1 = \varnothing, \\ 0 & \text{otherwise} \end{cases}$ for some $\psi_2 \in \mathcal{FH}_{\text{ex}}(\Gamma_2)$.

Note that in the first case we can recover these u_1, u_2, ψ_1, and ψ_2 as follows

$$u_1 = \sum_{|\lambda_1|=1} \dim(\lambda_1)\psi(\lambda_1, \varnothing), \qquad u_2 = 1 - u_1,$$

$$\psi_1(\lambda_1) = \frac{1}{u_1^{|\lambda_1|}} \cdot \psi(\lambda_1, \varnothing), \qquad \psi_2(\lambda_2) = \frac{1}{u_2^{|\lambda_2|}} \cdot \psi(\varnothing, \lambda_2).$$

The second and third cases above can be considered as a part of the first one with $u_1 = 1$ and $u_1 = 0$ respectively. Then we have a natural map from $\mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$ to

$$M := [0,1] \times \mathcal{FH}_{\text{ex}}(\Gamma_1) \times \mathcal{FH}_{\text{ex}}(\Gamma_2) \big/ \sim,$$

where $(u_1, \psi_1, \psi_2) \sim (\widetilde{u_1}, \widetilde{\psi_1}, \widetilde{\psi_2})$ if and only if either $u_1 = \widetilde{u_1} = 0$, $\psi_2 = \widetilde{\psi_2}$ or $u_1 = \widetilde{u_1} = 1$, $\psi_1 = \widetilde{\psi_1}$. It is readily seen that the map $\mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2) \to M$ is injective and continuous.

Keeping all the previous discussion in mind we multiply (4.9) by $\dim(\lambda_1)$ and sum over all λ_1 liying on a common level N of the graph Γ_1, and use the fact that

$$\sum_{|\lambda_1|=N} \dim(\lambda_1)\psi_1(\lambda_1) = 1.$$

Next we do the same thing for λ_2. Then we see by de Finetti's theorem, see Theorems 5.1 and 5.2 in [2], that the projection of the measure P to the first coordinate is concentrated at the point w_1, hence the integration in (4.9) must run over the set of harmonic functions of the first type 1). Then the factors $w_1^{|\lambda_1|} w_2^{|\lambda_2|}$ and $u_1^{|\lambda_1|} u_2^{|\lambda_2|}$ in the resulting expression cancel out, and, with some abuse of notation, we can write

$$\varphi_1(\lambda_1)\varphi_2(\lambda_2) = \int\limits_{\mathcal{FH}_{\text{ex}}(\Gamma_1) \times \mathcal{FH}_{\text{ex}}(\Gamma_2)} \psi_1(\lambda_1)\psi_2(\lambda_2) P(d\psi). \tag{4.10}$$

Taking $\lambda_2 = \varnothing$ in (4.10), we see that the measure P must be concentrated only at φ_1. Analogously, P is concentrated only at φ_2. Thus, P is a delta measure on $\mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$. $\square$

Next proposition follows immediately from the proof of Lemma 4.2.13.

Proposition 4.2.14. *Consider the space*

$$[0,1] \times \mathcal{FH}_{\text{ex}}(\Gamma_1) \times \mathcal{FH}_{\text{ex}}(\Gamma_2) \big/ \sim,$$

where the equivalence relation $\sim$ is defined by $(w, \varphi_1, \varphi_2) \sim (\widetilde{w}, \widetilde{\varphi_1}, \widetilde{\varphi_2})$ if and only if either $w = \widetilde{w} = 0$, $\varphi_2 = \widetilde{\varphi_2}$ or $w = \widetilde{w} = 1$, $\varphi_1 = \widetilde{\varphi_1}$.

The following maps

$$[0,1] \times \mathcal{FH}_{\text{ex}}(\Gamma_1) \times \mathcal{FH}_{\text{ex}}(\Gamma_2) \big/ \sim \longrightarrow \mathcal{FH}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$$

and

$$(0,1) \times \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1) \times \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_2) \longrightarrow \mathcal{FH}^{\circ}_{\text{ex}}(\Gamma_1 \times \Gamma_2),$$

defined by $(w, \varphi_1, \varphi_2) \mapsto \varphi$, where φ is given by (4.5) *with $w_1 = w$ and $w_2 = 1 - w$, are homeomorphisms.*

4.3 Semifinite harmonic functions

Our main goal is to prove the following semifinite analog of Theorem 4.2.10.

Theorem 4.3.1. *Let Γ_1 and Γ_2 be graded graphs and $\varphi \in \mathcal{H}_{\text{ex}}(\Gamma_1 \times \Gamma_2)$, then only one of the following situations can occur:*

1) *There exist $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ and real positive numbers w_1, w_2 with $w_1 + w_2 = 1$ such that*
$$\varphi(\lambda, \mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda) \varphi_2(\mu). \tag{4.11}$$
Moreover, these φ_1 and φ_2 are defined uniquely up to multiplicative constants.
We agree that $0 \cdot (+\infty) = 0$.

2) *There exist $\varphi_1 \in \mathcal{H}_{\text{ex}}(\Gamma_1)$ and $\nu_2 \in \Gamma_2$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \mu \not\leq \nu_2, \\ +\infty, & \text{if } \mu < \nu_2, \\ \varphi_1(\lambda), & \text{if } \mu = \nu_2. \end{cases} \tag{4.12}$$

3) *There exist $\nu_1 \in \Gamma_1$ and $\varphi_2 \in \mathcal{H}_{\text{ex}}(\Gamma_2)$ such that*
$$\varphi(\lambda, \mu) = \begin{cases} 0, & \text{if } \lambda \not\leq \nu_1, \\ +\infty, & \text{if } \lambda < \nu_1, \\ \varphi_2(\mu), & \text{if } \lambda = \nu_1. \end{cases} \tag{4.13}$$

Furthermore, every harmonic function on $\Gamma_1 \times \Gamma_2$ of the form 1), 2), or 3) is finite or semifinite, and indecomposable.

The ideal of a graded graph Γ generated by a vertex λ will be denoted by Γ^{λ}, i.e. $\Gamma^{\lambda} = \{\mu \in \Gamma \mid \mu \geq \lambda\}$. Such ideals will be called *principal* ideals.

Lemma 4.3.2. *Let Γ_1 and Γ_2 be primitive graded graphs. If $\varphi \in \mathcal{H}^\circ_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$, then there exist $\varphi_1 \in \mathcal{H}^\circ_{\mathrm{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}^\circ_{\mathrm{ex}}(\Gamma_2)$ and unique real positive numbers w_1, w_2 with $w_1 + w_2 = 1$ such that*

$$\varphi(\lambda,\mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda)\varphi_2(\mu). \tag{4.14}$$

Moreover, φ_1 and φ_2 are defined uniquely up to multiplicative constants.

Proof. Let us show that, if φ is given by (4.14), then numbers w_1, w_2 are uniquely defined and φ_1, φ_2 are uniquely defined up to multiplicative constants. Assume that

$$\varphi(\lambda,\mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1(\lambda)\varphi_2(\mu) = \widetilde{w_1}^{|\lambda|} \widetilde{w_2}^{|\mu|} \widetilde{\varphi_1}(\lambda)\widetilde{\varphi_2}(\mu). \tag{4.15}$$

The finiteness ideal of φ_i coincides with that of $\widetilde{\varphi_i}$, $i = 1,2$. Then we will assume that all further considerations will be performed inside the finiteness ideal of φ, which is equal to the direct product of finiteness ideals of φ_1 and φ_2. It makes possible to rewrite (4.15) as

$$\left(\frac{w_1}{\widetilde{w_1}}\right)^{|\lambda|} \frac{\varphi_1(\lambda)}{\widetilde{\varphi_1}(\lambda)} = \left(\frac{\widetilde{w_2}}{w_2}\right)^{|\mu|} \frac{\widetilde{\varphi_2}(\mu)}{\varphi_2(\mu)}. \tag{4.16}$$

The left-hand side of (4.16) depends only on λ, but not on μ and the right-hand side depends only on μ, but not on λ. Then both sides of (4.16) are constant, hence

$$\widetilde{\varphi_1}(\lambda) = c_1 \cdot \left(\frac{w_1}{\widetilde{w_1}}\right)^{|\lambda|} \varphi_1(\lambda) \tag{4.17}$$

$$\widetilde{\varphi_2}(\mu) = c_2 \cdot \left(\frac{w_2}{\widetilde{w_2}}\right)^{|\mu|} \varphi_2(\mu)$$

for some real positive constants c_1, c_2.

Next, $\widetilde{\varphi_1}$ is a harmonic function with respect to the multiplicity function $\varkappa_1$, but on the right-hand side of (4.17) we have a harmonic function with respect to the multiplicity function $\frac{w_1}{\widetilde{w_1}} \cdot \varkappa_1$. Hence $\widetilde{w}_1 = w_1$. Thus, φ_1 and $\widetilde{\varphi_1}$ are proportional. Similarly we show that $\widetilde{w}_2 = w_2$ and that φ_2 and $\widetilde{\varphi_2}$ are proportional.

Let $\varphi \in \mathcal{H}^\circ_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$ and take any $(\lambda,\mu) \in \Gamma_1 \times \Gamma_2$ such that $\varphi(\lambda,\mu) < +\infty$. This pair (λ,μ) will be fixed till the end of the proof. By Theorem 4.2.10 we have

$$\frac{\varphi(\nu_1,\nu_2)}{\varphi(\lambda,\mu)} = w_1^{|\nu_1|-|\lambda|} w_2^{|\nu_2|-|\mu|} \psi_1(\nu_1)\psi_2(\nu_2)$$

for any $\nu_1 \geq \lambda$, $\nu_2 \geq \mu$, some numbers $w_1, w_2 \in \mathbb{R}_{>0}$ with $w_1 + w_2 = 1$, and some finite strictly positive normalized harmonic functions ψ_1 and ψ_2 on Γ_1^λ and Γ_2^μ.

Let us denote by φ_1 and φ_2 the extensions of ψ_1 and ψ_2 to primitive graded graphs Γ_1 and Γ_2 respectively provided by Theorem 3.3.14. We will show that for any $\nu_1 \in \Gamma_1$ and any $\nu_2 \in \Gamma_2$

$$\varphi(\nu_1,\nu_2) \geq \varphi(\lambda,\mu) \cdot w_1^{|\nu_1|-|\lambda|} w_2^{|\nu_2|-|\mu|} \varphi_1(\nu_1)\varphi_2(\nu_2), \tag{4.18}$$

Then the claim follows from the indecomposability of φ.

First, we write

$$\varphi(\nu_1,\nu_2) = \lim_{N\to+\infty} \sum_{\substack{\lambda_1\in\Gamma_1,\lambda_2\in\Gamma_2:\\ |\lambda_1|+|\lambda_2|=N}} \dim\Big((\nu_1,\nu_1),(\lambda_1,\lambda_2)\Big)\varphi(\lambda_1,\lambda_2) =$$

$$\lim_{N\to+\infty} \sum_{\substack{n_1,n_2:\\ n_1+n_2=N}} \sum_{\substack{\lambda_1\in\Gamma_1,\lambda_2\in\Gamma_2:\\ |\lambda_1|=|\nu_1|+n_1,\\ |\lambda_2|=|\nu_2|+n_2}} \binom{N}{n_1} \dim_1(\nu_1,\lambda_1)\dim_2(\nu_2,\lambda_2)\varphi(\lambda_1,\lambda_2).$$

Omitting all terms except those for which $\lambda_1 \geq \lambda$ and $\lambda_2 \geq \mu$, we obtain

$$\varphi(\nu_1,\nu_2) \geq \varphi(\lambda,\mu)\cdot w_1^{|\nu_1|-|\lambda|} w_2^{|\nu_2|-|\mu|} \times$$
$$\lim_{N\to+\infty}\left[\sum_{\substack{n_1,n_2:\\ n_1+n_2=N}} \binom{N}{n_1} w_1^{n_1} w_2^{n_2} \sum_{\substack{\lambda_1\in\Gamma_1^{\lambda}:\\ |\lambda_1|=|\nu_1|+n_1}} \dim_1(\nu_1,\lambda_1)\psi_1(\lambda_1) \sum_{\substack{\lambda_2\in\Gamma_2^{\mu}:\\ |\lambda_2|=|\nu_2|+n_2}} \dim_2(\nu_2,\lambda_2)\psi_2(\lambda_2)\right].$$

Next, we omit all summands except those which satisfy $|n_1 - w_1 N| < N^{2/3}$. Note that the expressions $\sum_{\substack{\lambda_1\in\Gamma_1^{\lambda}:\\ |\lambda_1|=|\nu_1|+n_1}} \dim_1(\nu_1,\lambda_1)\psi_1(\lambda_1)$ and $\sum_{\substack{\lambda_2\in\Gamma_2^{\mu}:\\ |\lambda_2|=|\nu_2|+n_2}} \dim_2(\nu_2,\lambda_2)\psi_2(\lambda_2)$ are non-increasing in n_1 and n_2. Hence we can bound $\varphi(\nu_1,\nu_2)$ from below

$$\varphi(\nu_1,\nu_2) \geq \varphi(\lambda,\mu)\cdot w_1^{|\nu_1|-|\lambda|} w_2^{|\nu_2|-|\mu|} \lim_{N\to+\infty}\left[\sum_{\substack{n_1,n_2:\\ n_1+n_2=N\\ |n_1-w_1N|<N^{2/3}}} \binom{N}{n_1} w_1^{n_1} w_2^{n_2}\right.$$
$$\left.\sum_{\substack{\lambda_1\in\Gamma_1^{\lambda}:\\ |\lambda_1|=|\nu_1|+\lfloor w_1N-N^{2/3}\rfloor}} \dim_1(\nu_1,\lambda_1)\psi_1(\lambda_1) \sum_{\substack{\lambda_2\in\Gamma_2^{\mu}:\\ |\lambda_2|=|\nu_2|+\lfloor w_2N-N^{2/3}\rfloor}} \dim_2(\nu_2,\lambda_2)\psi_2(\lambda_2)\right],$$

where $\lfloor\cdot\rfloor$ is the floor function.

Finally, by the central limit theorem we have

$$\lim_{N\to+\infty} \sum_{\substack{n_1,n_2:\\ n_1+n_2=N\\ |n_1-w_1N|<N^{2/3}}} \binom{N}{n_1} w_1^{n_1} w_2^{n_2} = 1.$$

Thus, (4.18) follows immediately from the very definition of φ_1 and φ_2 and their strict positivity. □

Now we would like to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Let us check that expressions on the right hand sides of (4.11), (4.12), and (4.13) define finite or semifinite harmonic functions on $\Gamma_1\times\Gamma_2$. Functions from (4.12) and (4.13) are from $\mathcal{H}_{\text{ex}}(\Gamma_1\times\Gamma_2)$ by Boyer's lemma, see the proof below. To see that the function from (4.11) is in $\mathcal{H}_{\text{ex}}(\Gamma_1\times\Gamma_2)$ note that a harmonic function Φ is semifinite if and only if it is not finite and the following identity holds (see [24, Proposition 3.7])

$$\Phi(\lambda) = \lim_{N\to\infty} \sum_{\substack{\mu:\ \mu\geq\lambda,\ |\mu|=N\\ 0<\varphi(\mu)<+\infty}} \dim(\lambda,\mu)\Phi(\mu). \tag{4.19}$$

To prove this equality for the function from (4.11) we use an argument with the central limit theorem similar to that which was used in the proof of Lemma 4.3.2. Namely, we assume that $\varphi(\lambda,\mu) = +\infty$ and bound the right hand side of the equality (4.19) from below to see that it equals $+\infty$.

It is clear that the functions defined by (4.12) and (4.13) are indecomposable, if the functions φ_1 and φ_2 are. We will check that the function from (4.11) is neccesarily indecomposable, if φ_1 and φ_2 are indecomposable. For this we note that $\mathrm{supp}(\varphi) = \mathrm{supp}(\varphi_1) \times \mathrm{supp}(\varphi_2)$ and the direct product of two primitive graded graphs is primitive as well. Thus, by Proposition 3.3.16 $\mathrm{supp}(\varphi)$ is a primitive graded graph. It is enough to check that φ is indecomposable being restricted to $\mathrm{supp}(\varphi)$, hence further considerations will be performed inside $\mathrm{supp}(\varphi)$. From Theorem 3.3.14 and Lemma 4.2.13 applied to any principal ideal of $\mathrm{supp}(\varphi)$ that lies in the finiteness ideal of φ, it follows that φ is indecomposable being restricted to any principal ideal of $\mathrm{supp}(\varphi)$ that lies in the finiteness ideal. Recall that a principal ideal is an ideal generated by some vertex μ, i.e $\{\lambda \mid \lambda \geq \mu\}$. Thus, if $\varphi \geq \psi$, then $\psi = c_I \cdot \varphi$ on any principal ideal $I \subset \mathrm{supp}(\varphi)$ that lies in the finiteness ideal of φ, where c_I is a positive constant. Finally, we remark that in a primitive graph every two ideals have a non-empty intersection. Thus, c_I does not depend on I and φ is indecomposable on $\mathrm{supp}(\varphi)$.

Now we will prove that every harmonic function $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$ is of the form (4.11), (4.12) or (4.13). Note that, if $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$, then by Proposition 3.3.16 and Lemma 4.2.3 the support $\mathrm{supp}\,\varphi$ is of the form $J_1 \times J_2$, where either J_1 and J_2 are primitive saturated coideals of Γ_1 and Γ_2 respectively, or one of them is saturated and primitive and another one is principal. The first case of the theorem corresponds to the first case mentioned above and follows from Lemma 4.3.2 immediately. Let us assume that $J_1 \subseteq \Gamma_1$ is a saturated primitive coideal and $J_2 \subset \Gamma_2$ is a principal coideal corresponding to some vertex $\nu_2 \in \Gamma_2$. Then $\varphi(\lambda, \mu) = 0$ if $\mu \not\leq \nu_2$. Consider the ideal of $J_1 \times J_2$ consisting of all pairs $(\lambda, \nu_2) \in J_1 \times J_2$, whose second component equals exactly ν_2. Obviously, it is isomorphic to J_1 as a Bratteli diagram. Let us denote this ideal by $(J_1 \times J_2)^{(-,\nu_2)}$. Then Theorem 3.3.14 provides us a bijection between strictly positive finite and semifinite indecomposable harmonic functions on $J_1 \times J_2$ and on $(J_1 \times J_2)^{(-,\nu_2)} \simeq J_1$. The last thing which remains to show is to indicate how we should extend harmonic functions from the ideal $(J_1 \times J_2)^{(-,\nu_2)}$ to the whole graph $J_1 \times J_2$. If $J_2 = \{\nu_2\}$ is a singleton, then the extension is trivial, otherwise it is sufficient to take any ν' such that $\nu' \nearrow \nu_2$ and prove that $\varphi(\lambda, \nu') = +\infty$, $\lambda \in J_1$. Consider the ideal $(J_1 \times J_2)^{(-,\nu')}$ of $J_1 \times J_2$ consisting of all pairs $(\lambda, \mu) \in J_1 \times J_2$, whose second component μ is greater than or equal to ν', that is $\mu = \nu'$ or $\mu = \nu_2$. Then

$$(J_1 \times J_2)^{(-,\nu')} = J_1 \times \{\nu'\} \sqcup J_1 \times \{\nu_2\},$$

and $J_1 \times \{\nu_2\}$ is an ideal of $(J_1 \times J_2)^{(-,\nu')}$, and there is a natural map $J_1 \times \{\nu'\} \to J_1 \times \{\nu_2\}$, $(\lambda, \nu') \mapsto (\lambda, \nu_2)$ which, for a trivial reason, is an isomorphism of Bratteli diagrams. Take $\lambda \in J_1$. Using the notation from Lemma 3.5.3, we set $\Gamma = (J_1 \times J_2)^{(-,\nu')}$, $I = J_1 \times \{\nu_2\}$, $\beta_{(\lambda,\nu')} = \varkappa_2(\nu', \nu_2)$ and consider (λ, ν_2) as λ' from the lemma. Then the left hand side of (3.7) consists of only one summand and the inequality turns into the equality, hence Lemma 3.5.3 implies that $\varphi(\lambda, \nu') = +\infty$ for any $\nu' \nearrow \nu_2$. By assumption φ is semifinite at (λ, ν_2), hence it is semifinite at (λ, ν') as well. □

Remark 4.3.3. One can deal with the cases 2) and 3) from Theorem 4.3.1 applying the Vershik-Kerov ergodic method, see [10, Theorem on p. 60]. This argument turns out to be simplier then that presented above. The reason why we use Boyer's lemma is that it guarantees that the functions defined by the right hand sides of (4.12) and (4.13) are finite or semifinite for any $\varphi_1 \in \mathcal{H}_{\mathrm{ex}}(\Gamma_1)$, $\varphi_2 \in \mathcal{H}_{\mathrm{ex}}(\Gamma_2)$.

4.3.1 Semifinite Vershik-Kerov ring theorem

There is a well known analog of the Vershik-Kerov ring theorem for semifinite indecomposable harmonic functions on a multiplicative branching graph Theorem 3.4.4, see [15, Theorem p.134], [7, Proposition 8.4], and [15, Theorem p.144]. We apply it to the case of the direct product of two multiplicative graphs (Corollary 4.3.4).

We can associate a *finite* indecomposable harmonic function to any harmonic function $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma)$ on a multiplicative graph Γ. We will denote it by φ^{fin}, i.e. $\varphi^{fin} = \varphi$, if φ is finite and $\varphi^{fin} = \psi$, where ψ is given by Theorem 3.4.4, if φ is semifinite.

Corollary 4.3.4. *Let* $(\Gamma_1, A, \{a_\lambda\}_{\lambda\in\Gamma_1})$ *and* $(\Gamma_2, B, \{b_\mu\}_{\mu\in\Gamma_2})$ *be multiplicative graphs, and* $\varphi \in \mathcal{H}_{\mathrm{ex}}(\Gamma_1 \times \Gamma_2)$. *Then*

- *in case 1) of Theorem 4.3.1*

$$\varphi^{fin}(\lambda,\mu) = w_1^{|\lambda|} w_2^{|\mu|} \varphi_1^{fin}(\lambda)\varphi_2^{fin}(\mu);$$

- *in case 2) of Theorem 4.3.1*

$$\varphi^{fin}(\lambda,\mu) = \begin{cases} 0, & \text{if } \mu \neq \varnothing, \\ \varphi_1^{fin}(\lambda), & \text{if } \mu = \varnothing; \end{cases}$$

- *in case 3) of Theorem 4.3.1*

$$\varphi^{fin}(\lambda,\mu) = \begin{cases} 0, & \text{if } \lambda \neq \varnothing, \\ \varphi_2^{fin}(\mu), & \text{if } \lambda = \varnothing. \end{cases}$$

4.4 Slow graphs and inverse symmetric semigroups

Let $\mathbb{N}$ be the representation of natural numbers as a half-line Bratteli diagram, with a single vertex on each level and a single edge between adjacent levels. Recall that for a graded graph Γ the corresponding *slow graph* can be defined as a product $\mathbb{N}\times\Gamma$. Such graphs were defined and studied in [30]. Their name comes from the following natural description of the path space for a slow graph. Informally, we take any path in the initial graph, and at each step we either move along the path, or stay at the same vertex once again. More formally, let $\mathcal{T}(\Gamma)$ stand for the space of infinite paths in Γ starting at $\varnothing$. We take any path $(\varnothing \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \dots) \in \mathcal{T}(\Gamma)$ and an increasing sequence of positive numbers $i_1 < i_2 < \cdots$, and construct a path

$$\Big((0,\varnothing) \nearrow (1,\varnothing) \nearrow \dots (i_1,\varnothing) \nearrow (i_1,\lambda_1) \nearrow (i_1+1,\lambda_1) \nearrow \dots \\ (i_2,\lambda_1) \nearrow (i_2,\lambda_2) \nearrow \dots\Big) \in \mathcal{T}(\mathbb{N}\times\Gamma).$$

Indecomposable finite harmonic functions are known for the slow graphs.

Theorem 4.4.1. *[30] Let Γ be a branching graph and Φ be a normalized indecomposable finite harmonic function on $\mathbb{N}\times\Gamma$, i.e. $\Phi \in \mathcal{FH}_{\mathrm{ex}}(\mathbb{N}\times\Gamma)$. Then only one of the following two situations can occur:*

1) *either there exist $\varphi \in \mathcal{FH}_{\mathrm{ex}}(\Gamma)$ and a real number w, $0 < w \le 1$, such that*

$$\Phi(n,\mu) = (1-w)^n w^{|\mu|} \varphi(\mu)$$

(and φ and w are uniquely defined);

2) *or $\Phi(n,\mu) = 1$ if $\mu = \varnothing$, and $\Phi(n,\mu) = 0$ otherwise.*

Furthermore, every such harmonic function on $\mathbb{N}\times\Gamma$ is indecomposable.

Remark 4.4.2. Theorem 4.4.1 easily follows from Theorem 4.2.10. The current statement differs slightly from the original one in [30] — the degeneration of the harmonic functions for $w = 0$ was not mentioned there. Namely the case 2) in Theorem 4.4.1 can be obtained from 1) if we set $w = 0$, but in this case there is no dependence on φ. Moreover, the notation in [30] differs slightly from the notation in the present paper. Namely, there the vertices of the n-th level of the corresponding slow graph were denoted by $\{(n,\lambda)|\lambda\in\Gamma,|\lambda|\le n\}$ instead of $(\mathbb{N}\times\Gamma)_n = \{(k,\lambda)|\lambda\in\Gamma,k+|\lambda| = n\}$.

Indecomposable semifinite harmonic functions for a slow graph can be easily described as a corollary to Theorem 4.3.1. Some simplifications are due to the fact that there is only one (trivial) normalized indecomposable finite harmonic function on $\mathbb{N}$ and no semifinite ones.

Corollary 4.4.3. *Let Γ be a graded graph and Φ be an indecomposable semifinite harmonic function on $\mathbb{N}\times\Gamma$, then only one of the following situations can occur:*

1) There exist an indecomposable semifinite harmonic function φ on Γ and a real number w, $0 < w \le 1$ such that

$$\Phi(n,\mu) = (1-w)^n w^{|\mu|}\varphi(\mu).$$

Moreover, φ and w are uniquely defined.

2) There exist $\nu\in\Gamma$, $\nu\ne\varnothing$ and a real positive number c such that

$$\Phi(n,\mu) = \begin{cases} 0, & \text{if } \mu\not\le\nu, \\ +\infty, & \text{if } \mu<\nu, \\ c, & \text{if } \mu=\nu. \end{cases}$$

3) There exist $m\in\mathbb{N}$, $m\ge 1$ and $\varphi\in\mathcal{H}_{\mathrm{ex}}(\Gamma)$ such that

$$\Phi(n,\mu) = \begin{cases} 0, & \text{if } n>m, \\ +\infty, & \text{if } n<m, \\ \varphi(\mu), & \text{if } n=m. \end{cases}$$

Furthermore, every such harmonic function on $\mathbb{N}\times\Gamma$ is semifinite and indecomposable.

A natural example of a slow graph is given by the representation theory of *inverse symmetric semigroups R_n*. The semigroup R_n can be defined as the semigroup of partial bijections of the set $\{1,2,\dots,n\}$, and it naturally contains the symmetric group S_n (the group of bijections of the same set $\{1,2,\dots,n\}$). Recall that the Bratteli diagram correspodning to the chain of the group algebras $\{\mathbb{C}[S_n]\}$ is the Young graph $\mathbb{Y}$ (see e.g. [20]). The semigroup algebras $\mathbb{C}[R_n]$ are semisimple, and the Bratteli diagram correspodning to the chain of the semigroup algebras $\{\mathbb{C}[R_n]\}$ is given by the slow graph $\mathbb{N}\times\mathbb{Y}$ (see [8]). Description of the semifinite harmonic functions in this case was one of the motivations for the present section.

Recall that indecomposable finite normalized harmonic functions on $\mathbb{Y}$ are given by the celebrated Thoma's theorem [29, 13], and semifinite traces on $\mathbb{Y}$ were described in [13, Theorem 3 on p.27] and [31, Theorem 9 on p.150].

Theorem 4.4.4. *[29, 13] Every indecomposable finite normalized harmonic function on the Young graph $\mathbb{Y}$ is of the form $\varphi_{\alpha,\beta}$, where α and β are sequences of non-increasing real non-negative numbers $\alpha = (\alpha_1\ge\alpha_2\ge\ldots\ge 0)$ and $\beta = (\beta_1\ge\beta_2\ge\ldots\ge 0)$ subject to $\sum_{i=1}^{\infty}\alpha_i + \sum_{j=1}^{\infty}\beta_j \le 1$.*

The function $\varphi_{\alpha,\beta}$ is defined as follows $\varphi_{\alpha,\beta}(\lambda) = s^\circ_\lambda(\alpha|\beta)$, where $s^\circ_\lambda(\alpha|\beta)$ is the image of the Schur function s_λ under the map $Sym \to \mathbb{R}$, $p_1 \mapsto 1$, $p_n \mapsto \sum_{i=1}^{\infty} \alpha_i^n + (-1)^{n-1} \sum_{j=1}^{\infty} \beta_j^n$ for $n \geq 2$.

Theorem 4.4.5. *[13, Theorem 3 on p.27, 31, Theorem 9 on p.150] Every indecomposable semifinite harmonic function on the Young graph $\mathbb{Y}$ is proportional to some $\varphi^\nu_{\alpha,\beta}$, where ν is a non-empty Young diagram and α and β are tuples of non-increasing real positive numbers $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k > 0)$ and $\beta = (\beta_1 \geq \beta_2 \geq \ldots \geq \beta_l > 0)$ subject to $\sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{l} \beta_j = 1$. The function $\varphi^\nu_{\alpha,\beta}$ is defined as follows*

$$\varphi^\nu_{\alpha,\beta}(\lambda) = \begin{cases} 0, & \text{if } \lambda \notin \mathbb{Y}^\nu_{k,l}, \\ +\infty, & \text{if } \lambda \in \mathbb{Y}^\nu_{k,l} \text{ but } \lambda \text{ does not cover the flange } \nu, \\ \varphi_{\alpha,\beta}(\lambda^f), & \text{if } \lambda \in \mathbb{Y}^\nu_{k,l} \text{ and } \lambda \text{ covers the flange } \nu, \end{cases}$$

where

- *$\mathbb{Y}^\nu_{k,l}$ is the coideal of the Young graph formed by all Young diagrams that can be fitted into the infinite hook consisting of k infinite rows and l infinite columns with an added flange of the form ν to the corner of the hook, see Figure 4.1;*
- *$\varphi_{\alpha,\beta}$ is the finite indecomposable harmonic function on $\mathbb{Y}$ associated to (α, β);*
- *$\lambda^f = \lambda - \nu$ is the Young diagram λ with the flange ν removed.*

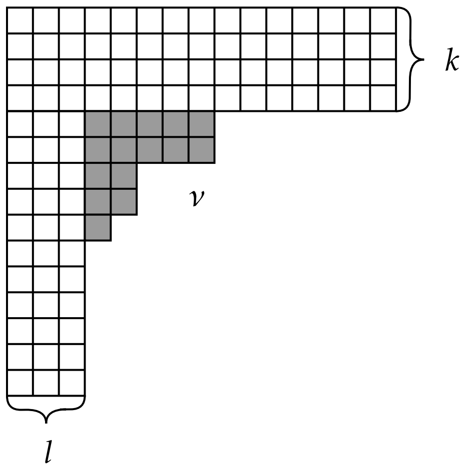


Figure. 4.1: An example of $\mathbb{Y}^\nu_{k,l}$ for $k = 4$, $l = 3$ and $\nu = (5, 5, 2, 2, 1)$.
White rows and columns represent infinite rows and columns.

Now we can easily combine Theorem 4.4.4 and Theorem 4.4.5 with Corollary 4.4.3 to describe semifinite harmonic functions on $\mathbb{N} \times \mathbb{Y}$.

Proposition 4.4.6. *Each semifinite indecomposable harmonic function on $\mathbb{N} \times \Gamma$ is proportional to $\Phi^\nu_{\alpha,\beta,w}$, $\Phi^k_{\alpha,\beta}$, $\Phi^{\nu,k}_{\alpha,\beta}$, or Φ^μ, where*

1) $\Phi^\nu_{\alpha,\beta,w}$ is defined by

$$\Phi^\nu_{\alpha,\beta,w}(n, \lambda) = (1 - w)^n w^{|\lambda|} \varphi^\nu_{\alpha,\beta}(\lambda)$$

for some $w \in (0, 1]$ and a semifinite indecomposable harmonic function $\varphi^\nu_{\alpha,\beta}$ on the Young graph $\mathbb{Y}$ from Theorem 4.4.5;

2) $\Phi^{k}_{\alpha,\beta}$ *and* $\Phi^{\nu,k}_{\alpha,\beta}$ *are defined by*

$$\Phi^{k}_{\alpha,\beta}(n,\lambda)=\begin{cases}0, & \text{if } n>k,\\ +\infty, & \text{if } n<k,\\ \varphi_{\alpha,\beta}(\lambda), & \text{if } n=k,\end{cases}\qquad \Phi^{\nu,k}_{\alpha,\beta}(n,\lambda)=\begin{cases}0, & \text{if } n>k,\\ +\infty, & \text{if } n<k,\\ \varphi^{\nu}_{\alpha,\beta}(\lambda), & \text{if } n=k,\end{cases}$$

for some $\varphi_{\alpha,\beta}$ *from Theorem 4.4.4,* $\varphi^{\nu}_{\alpha,\beta}$ *from Theorem 4.4.5, and an integer* $k>0$*;*

3) Φ^{μ} *is defined by*

$$\Phi^{\mu}(n,\lambda)=\begin{cases}0, & \text{if } \lambda\not\leq\mu,\\ +\infty, & \text{if } \lambda<\mu,\\ 1, & \text{if } \lambda=\mu,\end{cases}$$

for some $\mu\in\mathbb{Y}$, $\mu\neq\varnothing$.

4.5 Inverse of the map from Proposition 4.2.14

This section is devoted to one simple (and almost elementary) fact about finite harmonic functions on the direct product of two branching graphs, see Proposition 4.5.1 and Proposition 4.5.3 below. From now on we consider arbitrary finite harmonic functions on the direct product of two branching graphs, but not only indecomposable ones.

Let Γ_1 and Γ_2 be branching graphs, φ_1, φ_2 be finite normalized harmonic functions on them and let $w_1,w_2\in\mathbb{R}_{>0}$ be such that $w_1+w_2=1$. Then

$$\varphi(\lambda_1,\lambda_2)=w_1^{|\lambda_1|}w_2^{|\lambda_2|}\varphi_1(\lambda_1)\varphi_2(\lambda_2) \tag{4.20}$$

is a finite normalized harmonic function on $\Gamma_1\times\Gamma_2$[2].

Proposition 4.5.1. *Keeping the aforementioned notation we can recover* φ_1, φ_2, w_1 *and* w_2 *from* φ *by the following formulas*

$$w_1^{k_1}w_2^{k_2}=\sum_{|\lambda_1|=k_1,|\lambda_2|=k_2}\dim_1(\lambda_1)\dim_2(\lambda_2)\varphi(\lambda_1,\lambda_2), \tag{4.21}$$

$$\varphi_1(\lambda_1)=\sum_{n_2=0}^{\infty}\binom{n_2+|\lambda_1|-1}{n_2}\sum_{|\lambda_2|=n_2}\dim_2(\lambda_2)\varphi(\lambda_1,\lambda_2), \tag{4.22}$$

$$\varphi_2(\lambda_2)=\sum_{n_1=0}^{\infty}\binom{n_1+|\lambda_2|-1}{n_1}\sum_{|\lambda_1|=n_1}\dim_1(\lambda_1)\varphi(\lambda_1,\lambda_2). \tag{4.23}$$

Proof. Identities (4.22) and (4.23) reduce to

$$\frac{1}{(1-y)^{k+1}}=\sum_{n=0}^{\infty}\binom{n+k}{k}y^n \tag{4.24}$$

and (4.21) is obvious. □

Remark 4.5.2. Formulas (4.21), (4.22), and (4.23) may be viewed as an inverse of the map from Proposition 4.2.14. Compare them with the first two formulas from the proof of Theorem 2.8 in [30].

[2]Note that for any finite harmonic function f on the Pascal graph $\mathbb{P}$ the function φ defined by $\varphi(\lambda_1,\lambda_2)=f(|\lambda_1|,|\lambda_2|)\varphi_1(\lambda_1)\varphi_2(\lambda_2)$ is harmonic on $\Gamma_1\times\Gamma_2$.

Remark that the right hand side of (4.21) defines a harmonic function on the Pascal graph $\mathbb{P}$ for any finite normalized harmonic function φ on $\Gamma_1 \times \Gamma_2$. Proposition 4.5.1 shows that the right hand sides of (4.22) and (4.23) define harmonic functions on Γ_1 and Γ_2, if φ is of the form (4.20). In fact, more general claim holds.

Proposition 4.5.3. *The functions on Γ_1 and Γ_2 defined by the right hand sides of* (4.22) *and* (4.23) *are finite and harmonic for any finite normalized harmonic function φ on $\Gamma_1 \times \Gamma_2$.*

Proof 1. The statement is a trivial consequence of Proposition 4.2.14, Proposition 4.5.1, and Choquet's theorem [22, 21, Theorem 9.2]. □

Proof 2. One can prove this proposition using only elementary methods and de Finetti's Theorem, see Theorems 5.1 and 5.2 in [2]. The key observation is that the following expression

$$\sum_{|\lambda_1|=k_1, |\lambda_2|=k_2} \dim_1(\lambda_1)\dim_2(\lambda_2)\varphi(\lambda_1,\lambda_2) \tag{4.25}$$

defines a harmonic function on the Pascal graph $\mathbb{P}$ for any finite harmonic function φ on $\Gamma_1 \times \Gamma_2$. Then by Theorem 5.1 from [2] (4.25) is a mixture of indecomposables. Next, from the inequality

$$\dim_1(\lambda_1) \sum_{|\lambda_2|=n_2} \dim_2(\lambda_2)\varphi(\lambda_1,\lambda_2) \le \sum_{|\mu|=|\lambda_1|, |\lambda_2|=n_2} \dim_1(\mu)\dim_2(\lambda_2)\varphi(\mu,\lambda_2),$$

identity (4.24) and the integral representation of (4.25), it follows that the expression defined by the right hand side of (4.22) is finite and not exceeding $\dfrac{1}{\dim_1(\lambda_1)}$. Let us denote it by π_1, i.e.

$$\pi_1(\lambda) := \sum_{n_2=0}^{\infty} \binom{n_2+|\lambda|-1}{n_2} \sum_{|\lambda_2|=n_2} \dim_2(\lambda_2)\varphi(\lambda,\lambda_2) \le \frac{1}{\dim_1(\lambda)} < +\infty. \tag{4.26}$$

To establish the harmonicity condition for π_1 we prove the following identity by induction on k

$$\begin{aligned}\pi_1(\lambda) - \sum_{\mu \searrow \lambda} \varkappa_1(\lambda,\mu)\pi_1(\mu) = \sum_{n_2=k}^{\infty} \binom{n_2+|\lambda|-1-k}{|\lambda|-1} \sum_{|\lambda_2|=n_2} \dim_2(\lambda_2)\varphi(\lambda,\lambda_2) - \\ \sum_{\mu\searrow\lambda} \sum_{n_2=k}^{\infty} \binom{n_2+|\mu|-1-k}{|\mu|-1} \sum_{|\lambda_2|=n_2} \dim_2(\lambda_2)\varphi(\mu,\lambda_2)\end{aligned} \tag{4.27}$$

Finally, we recall that

$$\binom{n_2+|\lambda|-1-k}{|\lambda|-1} \le \binom{n_2+|\lambda|-1}{|\lambda|-1},$$

hence each of the summands in (4.27) tends to 0 as k goes to $+\infty$. □

Remark 4.5.4. Take φ as in (4.20), then $\psi_1(\lambda) := \varphi(\lambda,\varnothing)$ is a finite harmonic function on a branching graph $(\Gamma_1, \widetilde{\varkappa_1})$ that is similar to $(\Gamma_1, \varkappa_1)$ in the sense of Kerov, see Definition in §4 from [11]. The multiplicity function of the new graph differs from $\varkappa_1$ by w_1, see (4.20). Below we prove that the same is true for an arbitrary finite harmonic function on $\Gamma_1 \times \Gamma_2$.

We would like to generalize the key observation from the previous remark a bit. The claim is that the following expression defines a harmonic function on the Pascal graph $\mathbb{P}$ for any $\nu_1 \in \Gamma_1$, $\nu_2 \in \Gamma_2$, and any finite normalized harmonic function φ on $\Gamma_1 \times \Gamma_2$

$$\Pi^{\nu_1,\nu_2}(k_1,k_2) := \sum_{\substack{|\lambda_1|=|\nu_1|+k_1,\\ |\lambda_2|=|\nu_2|+k_2}} \dim_1(\nu_1,\lambda_1)\dim_2(\nu_2,\lambda_2)\varphi(\lambda_1,\lambda_2).$$

Next by Theorem 5.1 from [2] it is represented by a probability measure, say P^{ν_1,ν_2}, on $[0,1]$, that is

$$\Pi^{\nu_1,\nu_2}(k_1,k_2) = \varphi(\nu_1,\nu_2)\int_{[0,1]} w_1^{k_1} w_2^{k_2} P^{\nu_1,\nu_2}(dw).$$

Then by (4.24) the expression $\pi_1(\lambda)$ defined by (4.26) equals

$$\varphi(\lambda,\varnothing)\int_{[0,1]} w_1^{-|\lambda|} P^{\lambda,\varnothing}(dw).$$

Recall that it is finite for all $\lambda \in \Gamma_1$.

Thus, the harmonicity condition for π_1 implies that the function $\psi_1(\lambda) := \varphi(\lambda,\varnothing)$ is a finite harmonic function on a branching graph that is similar to $(\Gamma_1,\varkappa_1)$ in the sense of Kerov.

Chapter 5

The zigzag graph

5.1 Summary in French

Dans cette section, nous décrivons les fonctions harmoniques semi-finies sur le graphe en zigzag et prouvons un analogue semi-fini du théorème de l'anneau de Vershik-Kerov pour celui-ci.

Considérons les compositions (partitions ordonnées) de nombres naturels. Nous les identifions avec les diagrammes en ruban, qui sont des diagrammes de Young imbriqués connectés écrits selon la convention française et ne contenant pas de blocs de boîtes de taille 2×2. Une composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ est identifiée avec le diagramme de Young en ruban ayant λ_i boîtes dans la i-ème lignes. Par exemple, la seule composition de 1 est identifiée avec $\square$. Le nombre de boîtes dans λ est égal à $|\lambda| = \lambda_1 + \ldots + \lambda_l$. Nous considérons les diagrammes de Young en ruban comme des zigzags se déplaçant du coin supérieur gauche au coin inférieur droit. Il existe une bijection entre les zigzags et les mots binaires.

Un mot binaire est un mot dans l'alphabet de deux symboles, $+$ et $-$. Nous utiliserons les conventions suivantes

$$\overset{n}{+} = \underbrace{+\ldots+}_{n} \quad \text{et} \quad \overset{n}{-} = \underbrace{-\ldots-}_{n}.$$

La bijection entre les zigzags et les mots binaires est la suivante. De gauche à droite, nous lisons les symboles du mot binaire et ajoutons des cases au zigzag le plus simple $\square$. Si le symbole est $+$, nous ajoutons une case dans la direction horizontale vers la droite, et si le symbole est $-$, nous ajoutons une case dans la direction verticale vers le bas. Par exemple, le mot binaire $-+$ correspond au zigzag avec une case dans la première rangée et deux cases dans la deuxième rangée. Le mot binaire correspondant à un zigzag λ sera noté $\mathsf{bw}(\lambda)$. Ainsi, $\mathsf{bw}(\square)$ est le mot binaire vide.

Chaque mot binaire peut être représenté de manière unique comme une réunion consécutive de *blocs* avec des signes alternés. Par un *bloc*, nous entendons un uplet de symboles du même signe. Par exemple, le mot $+-\overset{3}{+}$ se divise en trois blocs, $+$, $-$ et $\overset{3}{+}$. Ainsi, un bloc peut être positif ou négatif en fonction du signe des symboles. En ce qui concerne les zigzags, ces blocs positifs et négatifs correspondent aux lignes et aux colonnes.

Par un *cluster*, nous entendons un symbole, $+$ ou $-$, auquel est associée une multiplicité positive formelle, qui peut être infinie. Nous disons qu'un cluster est *infini*, si sa multiplicité est infinie, sinon nous disons que le cluster est *fini*. Un *modèle* est une collection ordonnée de clusters alternés. De plus, nous supposons toujours qu'un modèle contient au moins un cluster infini.

Définition 5.1.1. Un modèle est appelé *fini*, s'il *ne contient pas* de clusters finis, à l'exception de ceux qui ne sont pas extrémaux et dont les deux voisins sont des clusters infinis du même signe. Un modèle qui n'est pas fini sera appelé *semi-fini*.

Soit t un modèle semi-fini. Par un *cluster séparant* de t, nous entendons un cluster d'un seul symbole qui n'est pas un cluster extrémal de t et dont les deux voisins sont des clusters infinis du même signe. Par le *flanc zigzag* de t, nous entendons un uplet de mots binaires dont chaque élément est composé de clusters finis mais non séparants de t se tenant à proximité. Le flanc zigzag sera noté $\mathsf{fl}(t)$.

Soit t un modèle semi-fini. Par une *section* de t, nous entendons une collection maximale de clusters consécutifs formant un modèle fini. Remarquons que les mots du flanc zigzag de t divisent t en sections.

Soit t un modèle arbitraire. Par t_n, nous désignons le mot binaire qui est obtenu à partir de t en remplaçant toutes les multiplicités infinies par le nombre naturel n. Alors le sous-ensemble du graphe en zigzag $\mathcal{Z}(t) := \{\lambda \in \mathcal{Z} \mid \mathsf{bw}(\lambda) < t_n \text{ pour un certain } n\}$ consiste en tous les zigzags (ou mots binaires) qui sont de *la forme* t.

Définition 5.1.2. Posons $J(t) = \bigcup_r \mathcal{Z}(r)$ pour un modèle semi-fini t, où la réunion est prise sur tous les r obtenus à partir de t en supprimant un seul symbole d'un cluster correspondant à un bloc d'un mot binaire du flanc zigzag $\mathsf{fl}(t)$.

Supposons que t ait k sections $t_1,\ldots,t_k$. Supposons que $\mathsf{fl}(t) = (a_0,\ldots,a_k)$ et que la partition de t soit de la forme:

$$t = (a_0,\ t_1,\ a_1,\ \ldots\ ,a_{k-1},\ t_k,\ a_k).$$

Si a_0 ou a_k est le mot binaire vide, nous l'ignorons dans tout ce qui suit.

Lemme 5.1.3. Si $\lambda \in \mathcal{Z}(t)\backslash J(t)$, alors

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k$$

pour certains $\lambda^{(i)} \in \mathcal{Z}(t_i)$, qui sont alors uniques.

Définition 5.1.4. Par un *modèle de croissance en zigzag semi-fini*, nous entendons une paire (t,w), où t est un gabarit semi-fini ayant m grappes infinies et $w = (w_1,\ldots,w_m)$ est un m-uplet de nombres réels positifs tels que $w_1 + \ldots + w_m = 1$.

Soit (t,w) un modèle de croissance en zigzag semi-fini. La partition de t en sections nous donne une partition de w

$$w = v_1 \sqcup \ldots \sqcup v_k,$$

où chaque v_i est un uplet de nombres réels provenant de $w = (w_1,\ldots,w_m)$ correspondant aux clusters infinis de t_i.

Définition 5.1.5. Pour tout $\lambda \in \mathcal{Z}$, nous définissons

$$\varphi_{t,w}(\lambda) = \begin{cases} F_{\lambda^{(1)}}(v_1)\cdot\ldots\cdot F_{\lambda^{(k)}}(v_k), \text{ si } \lambda \in \mathcal{Z}(t)\backslash J(t), \\ +\infty, \text{ si } \lambda \in J(t), \\ 0, \text{ si } \lambda \notin \mathcal{Z}(t). \end{cases}$$

où $\lambda \mapsto (\lambda^{(1)},\ldots,\lambda^{(k)})$ est l'application donnée par le Lemme 5.1.3 et $F_{\lambda^{(i)}}(v_i)$ est défini comme suit :

$$F_\mu(x_1,x_2,\ldots,x_n) = \sum x_1^{|\mu(1)|} x_2^{|\mu(2)|} \ldots x_n^{|\mu(n)|}, \tag{5.1}$$

où la somme est prise sur les décompositions de μ en n zigzags $\mu(1),\ldots,\mu(n)$ tels que $\mu(i)$ soit une ligne, si le nombre $x_i \in \{w_1,\ldots,w_m\}$ correspond à un cluster positif de t, et $\mu(i)$ soit une colonne, si x_i correspond à un cluster négatif de t. Notons que certains de ces $\mu(i)$ peuvent être vides.

Notous que l'expression (5.1) provient d'une application multiplicative $QSym \to \mathbb{R}$, voir la Section 5.4.1 sur la construction de Kerov.

Théorème 5.1.6.

1) Pour tout modèle de croissance en zigzag semi-fini (t, w), la fonction $\varphi_{t,w}$ est une fonction harmonique semi-finie indécomposable sur $\mathbb{Z}$.

2) Toute fonction harmonique semi-finie indécomposable sur $\mathbb{Z}$ est proportionnelle à $\varphi_{t,w}$ pour certains modèles de croissance en zigzag semi-finis (t, w).[1]

3) Les fonctions $\varphi_{t,w}$ sont distinctes pour des modèles de croissance en zigzag semi-finis distincts (t, w).

Maintenant, nous aimerions formuler le théorème de l'anneau de Vershik-Kerov semi-fini pour le graphe en zigzag. Pour cela, nous étendons nos fonctions harmoniques semi-finis sur $\mathbb{Z}$ à $\mathrm{span}_{\mathbb{R}_{\geq 0}}\{F_\lambda \mid \lambda \in \mathbb{Z}\} \subset QSym$, où $\{F_\lambda\}_{\lambda \in \mathbb{Z}}$ sont les fonctions quasi-symétriques fondamentales.

Théorème 5.1.7. Soit (t, w) un modèle de croissance en zigzag semi-fini. Pour tout $\mu \in \mathbb{Z}(t) \backslash J(t)$ et $\lambda \in \mathbb{Z}$, nous avons

$$\varphi_{t,w}\left(F_\lambda F_\mu\right) = \varphi_w(F_\lambda)\varphi_{t,w}(F_\mu),$$

où φ_w est la fonction harmonique finie sur $\mathbb{Z}$ définie par $\varphi_w(\lambda) = F_\lambda(w)$, voir la formule (5.1) ci-dessus.

5.2 Zigzag diagrams

In this section we recall a few notions on the zigzag graph [7], see also [28].

Let us consider compositions (ordered partitions) of natural numbers. We identify them with the ribbon diagrams, which are connected skew Young diagrams written in the French notation and containing no 2×2 blocks of boxes. A composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is identified with the ribbon Young diagram having λ_i boxes in the i-th row. For instance, the only one composition of 1 gets identified with $\square$. The number of boxes in λ equals $|\lambda| = \lambda_1 + \ldots + \lambda_l$. We treat ribbon Young diagrams as zigzags crawling from the top-left corner to the bottom-right corner. There is a bijection between the zigzags and the binary words, which we will discuss in details.

A binary word is a word in the alphabet of two symbols, + and −. We will use the following conventions

$$\overset{n}{+} = \underbrace{+\ldots+}_{n} \quad \text{and} \quad \overset{n}{-} = \underbrace{-\ldots-}_{n}.$$

The bijection between the zigzags and the binary words is as follows. From left to right we read the symbols off the binary word and add boxes to the simplest zigzag $\square$. If the symbol is +, then we add a box in the horizontal direction to the right, and if the symbol is −, then we add a box in the vertical direction to the bottom. For instance, the binary word −+ corresponds to the zigzag with one box in the first row and two boxes in the second row. The binary word corresponding to a zigzag λ will be denoted by $\mathrm{bw}(\lambda)$. So, $\mathrm{bw}(\square)$ is the empty binary word. The number of symbols in a binary word will be denoted by $|\cdot|$, so $|\mathrm{bw}(\lambda)| = |\lambda| - 1$.

[1] Notous que l'ensemble des zéros de $\varphi_{t,w}$ est toujours non vide. Cela est conforme au théorème de non-existence de Wassermann en raison du fait que le graphe en zigzag est multiplicatif et que $QSym$ ne contient aucun diviseur de zéro, voir le Théorème 3.1.12.

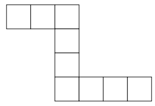
$\lambda =$  $\mathsf{bw}(\lambda) = + + - - - + + +$

Figure. 5.1: A zigzag diagram and its binary word

For binary words a and b we write $a \nearrow b$ if and only if $|b| = |a| + 1$ and a can be obtained from b by deleting a single symbol. For zigzags λ and μ we write $\lambda \nearrow \mu$ if and only if $\mathsf{bw}(\lambda) \nearrow \mathsf{bw}(\mu)$.

Definition 5.2.1. The zigzag graph $\mathbb{Z}$ is a graded graph $\mathbb{Z} = \bigsqcup_{n \geq 0} \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the set of all zigzags consisting of n boxes. By definition $\mathbb{Z}_0$ is a singleton $\mathbb{Z}_0 = \{\varnothing\}$. There is an edge going from λ to μ if and only if $\lambda \nearrow \mu$. All edges of $\mathbb{Z}$ are by definition simple.

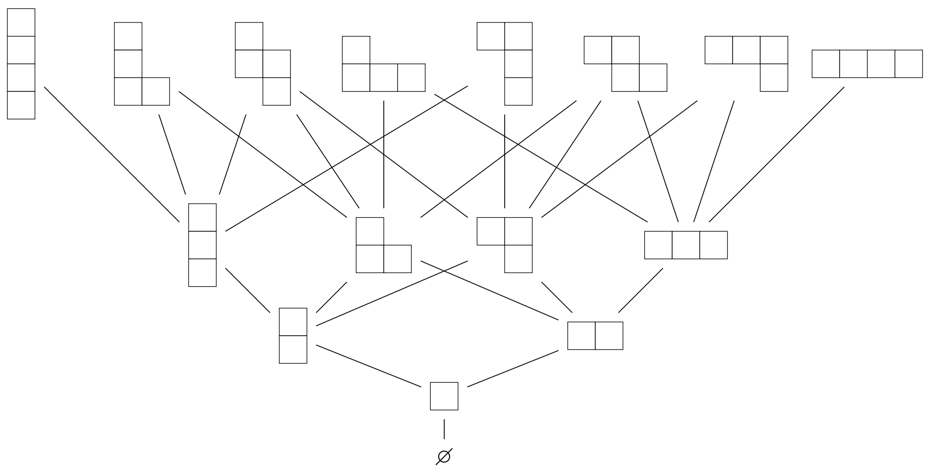


Figure. 5.2: The first few levels of the zigzag graph $\mathbb{Z}$

Let $QSym$ be the algebra of quasisymmetric functions and $\{F_\lambda\}_{\lambda \in \mathbb{Z}}$ be the fundamental quasisymmetric functions defined in [6], see also [25, p. 357]. There is a Pieri-type rule for this basis

$$F_\square F_\lambda = \sum_{\mu : \lambda \nearrow \mu} F_\mu,$$

which reflects the branching rule for the zigzag graph, see [25, p. 482, Exercise 7.93] or [17, p. 35, (3.13)].

5.3 Coideals of the zigzag graph

The algebra $QSym$ contains no zero divisors, since it is a subalgebra of the formal power series algebra in countably many variables. Then Theorem 3.4.5 implies that $\mathbb{Z}$ possesses no strictly positive indecomposable semifinite harmonic functions. From Proposition 3.3.16 it follows that the support of any indecomposable semifinite harmonic function on $\mathbb{Z}$ is a primitive saturated coideal. In this section we explicitly describe all primitive saturated coideals of $\mathbb{Z}$. Furthermore, we specify the coideals corresponding to the supports of the finite indecomposable harmonic functions. By Proposition 3.4.6 none of these coideals can be realised as the support of an indecomposable semifinite harmonic function.

5.3.1 Saturated coideals of the zigzag graph

Each binary word can be uniquely represented as a consecutive union of *blocks* with alternating signs. By a *block* we mean a tuple of symbols of the same sign. For instance, the word $+-\overset{3}{+}$ splits into three blocks, $+$, $-$, and $\overset{3}{+}$. So, a block can be positive or negative depending on the sign of symbols. As for zigzags, these positive and negative blocks correspond to rows and columns.

Definition 5.3.1. By a *cluster* we mean a symbol, $+$ or $-$, with an assigned to it formal positive multiplicity, which may be infinite. We say that a cluster is *infinite*, if its multiplicity is infinite, otherwise we say that the cluster is *finite*. A *template* is an ordered collection of alternating clusters. Furthermore, we always assume that a template contains at least one infinite cluster.

For instance, $\overset{\infty}{+}\overset{\infty}{-}\overset{3}{+}\overset{\infty}{-}$ is a template while $\overset{3}{+}\overset{2}{-}$ is not.

Each template can be thought of as an infinite zigzag consisting of finite number of possibly infinite rows and columns. Infinite rows and columns correspond to infinite clusters of this template. The infinite zigzag corresponding to the template t will be denoted by $\mathrm{z}(t)$, see Figure 5.3.

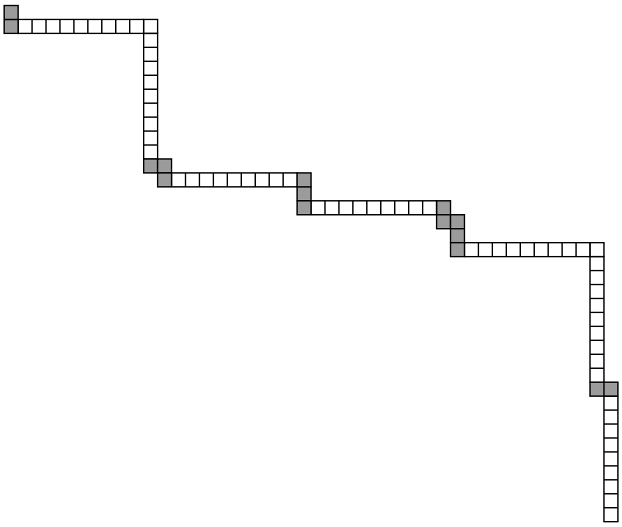


Figure. 5.3: The infinite zigzag $\mathrm{z}(t)$ for the template

$$t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{1}{-}\overset{\infty}{+}\overset{2}{-}\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{2}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}.$$

White strips of boxes represent infinitely long rows and columns while grey boxes represent zigzags corresponding to finite clusters of t.

To every template t we associate a coideal $\mathbb{Z}(t)$ of the zigzag graph, which is by definition of the form $\mathbb{Z}_\tau$ for some infinite path τ, see the paragraph above Proposition 3.2.6. In order to define this path τ, we replace infinite rows and columns in the infinite zigzag $\mathrm{z}(t)$ with long enough (but finite) rows and columns. So, we obtain a sequence of increasing zigzags. Then τ is any path in the zigzag graph that goes through all these zigzags. Equivalently, τ is any path that goes through all zigzags corresponding to the binary words obtained from t by replacing infinite clusters with long enough (but finite) blocks. Any such path τ completely "fills" the infinite zigzag $\mathrm{z}(t)$ that is, starting from some point, τ looks like a tuple of rows and columns, some of which grow infinitely large while others stay frozen; the frozen rows and columns correspond to finite clusters of t. Note that coideals $\mathbb{Z}(t_1)$ and $\mathbb{Z}(t_2)$ coincide if and only if templates t_1 and t_2 coincide. Below it will be useful sometimes to identify a template t with the corresponding coideal $\mathbb{Z}(t)$. Moreover, it will be convenient to view an infinite zigzag as a sequence of growing finite zigzags.

Proposition 5.3.2. *Each proper primitive saturated coideal of the zigzag graph is of the form $\mathbb{Z}(t)$ for some template t, which is uniquely defined.*

Proof. Let J be a proper saturated primitive coideal of the zigzag graph. From Proposition 3.2.6 it follows that there exists a path τ such that $J = \mathbb{Z}_\tau$. Since J is proper, it follows that the number of blocks in binary words corresponding to zigzags from the path τ is uniformly bounded along the path. Then we form a template t in the following way. Bounded blocks of binary words from τ correspond to finite clusters of t, and unbounded blocks correspond to infinite clusters. □

5.3.2 Templates and ideals of $QSym$

For a template t we consider the following linear subspace $I_t = \operatorname{span}_{\mathbb{R}}(F_\lambda \mid \lambda \notin \mathbb{Z}(t))$ of $QSym$. Note that I_t is a graded ideal of $QSym$, due to $F_\lambda F_\mu \in \operatorname{span}_{\mathbb{R}}(F_\nu \mid \nu \geq \lambda, \mu)$, see. [17, p. 35, (3.13)].

Observation 5.3.3. Let $t_1, \ldots, t_k$ be some templates. Then $\mathbb{Z}(t_1) \times \ldots \times \mathbb{Z}(t_k)$ is a multiplicative graph, which algebra is $\operatorname{span}_{\mathbb{R}}\left(F_{\lambda(1)} \otimes \ldots \otimes F_{\lambda(k)} \mid \lambda(i) \in \mathbb{Z}(t_i)\right) = {QSym}/{I_{t_1}} \otimes \ldots \otimes {QSym}/{I_{t_k}}$, see Definition 4.2.1.

5.4 Zero sets of finite harmonic functions

From Proposition 3.3.16 it follows that the support of a finite indecomposable harmonic function on the zigzag graph is a primitive coideal. If this coideal is proper, then by Proposition 5.3.2 it corresponds to a template. Below we specify which finite indecomposable harmonic functions have non-empty zero sets and for them we explicitly describe the corresponding templates.

5.4.1 Kerov's construction

Recall the definition of finitary oriented paintbox, see Definition 5.2 from [7].

Definition 5.4.1. A *finitary oriented paintbox* is a pair (w_+, w_-) of disjoint open subsets of the unit interval $(0, 1)$, each comprised of finitely many subintervals and such that the total Lebesgue measure of w_+ and w_- equals 1. The symbol W_0 stands for the set of all such pairs.

Lengths of intervals in $w = (w_+, w_-) \in W_0$ will be denoted by w_i. We agree that the intervals are ordered from left to right and w_1 denotes the length of the leftmost interval. We say that an interval of w is *positively oriented* if it belongs to w_+ and we say that an interval is *negatively oriented* if it belongs to w_-.

Kerov's construction produces a finite indecomposable harmonic function φ_w on the zigzag graph out of any finitary oriented paintbox $w \in W_0$, see [7, p. 13-18]. Let us briefly recall this procedure.

For any zigzag λ we set by definition

$$\varphi_w(\lambda) = F_\lambda(w),$$

where

$$F_\lambda(w) = (\psi_1 \otimes \ldots \otimes \psi_m) \circ \left(r_{w_1} \otimes \ldots \otimes r_{w_m}\right) \circ \Delta^{(m)}(F_\lambda), \;\; \lambda \in \mathbb{Z} \tag{5.2}$$

and

- m is the number of intervals in w;
- $\Delta^{(m)}\colon QSym \to QSym^{\otimes m}$ is the m-th iteration of the comultiplication Δ in $QSym$;
- r_t is the automorphism of the graded algebra $QSym$, defined by $r_t(F_\lambda) = t^{|\lambda|}F_\lambda$;
- $\psi_i = \psi_+$, if w_i is positively oriented and $\psi_i = \psi_-$, if w_i is negatively oriented, where

$$\psi_+(F_\lambda) = \begin{cases} 1, \text{ if } \lambda \text{ is a row,} \\ 0 \text{ otherwise,} \end{cases} \qquad \psi_-(F_\lambda) = \begin{cases} 1\text{ , if } \lambda \text{ is a column,} \\ 0 \text{ otherwise.} \end{cases}$$

Let us consider all splittings of the zigzag λ into m zigzags $\lambda(1),\dots,\lambda(m)$ such that $\lambda(i)$ is a row, if the interval w_i is positively oriented, and $\lambda(i)$ is a column, if w_i is negatively oriented. Note that some of these $\lambda(i)$ may be empty.

Proposition 5.4.2. *[7, Proposition 5.3] The following equality holds*

$$F_\lambda(w) = \sum w_1^{|\lambda(1)|} w_2^{|\lambda(2)|} \dots w_m^{|\lambda(m)|},$$

where the sum is taken over all splittings of λ mentioned above.

Let us denote by W the set of pairs of disjoint open subsets of the unit interval. Note that W_0 is a subset of W.

Theorem 5.4.3. *[7, Theorem 7.5] There is a bijective correspondence $w \mapsto \varphi_w$ between elements of $w \in W$ and indecomposable finite harmonic functions on the zigzag graph. For finitary oriented paintboxes this correspondence is defined by Kerov's construction.*

For any finitary oriented paintbox $w \in W_0$ we denote by t_w the template obtained from w by replacing positively and negatively oriented intervals with symbols $\overset{\infty}{+}$ and $\overset{\infty}{-}$ respectively and inserting between any two neighbor infinite symbols of the same type a symbol of the opposite type.

Proposition 5.4.4. *Let $w \in W$. Then* $\operatorname{supp}(\varphi_w) = \begin{cases} \mathcal{Z}(t_w), \textit{ if } w \in W_0, \\ \mathcal{Z}, \textit{ otherwise.} \end{cases}$

Proof. Suppose that the finitary oriented paintbox w consists of m intervals. Then Proposition 5.4.2 implies that $\varphi_w(\lambda) > 0$ if and only if λ can be represented as a consecutive union of m rows and columns taken in the order proposed by the orientations of intervals of w. Thus, $\varphi_w(\lambda) > 0$ if and only if $\lambda \in \mathcal{Z}(t_w)$.

Now let $w \in W \setminus W_0$. It suffices to show that $\varphi_w(\lambda_{2n}) > 0$ for any n, where

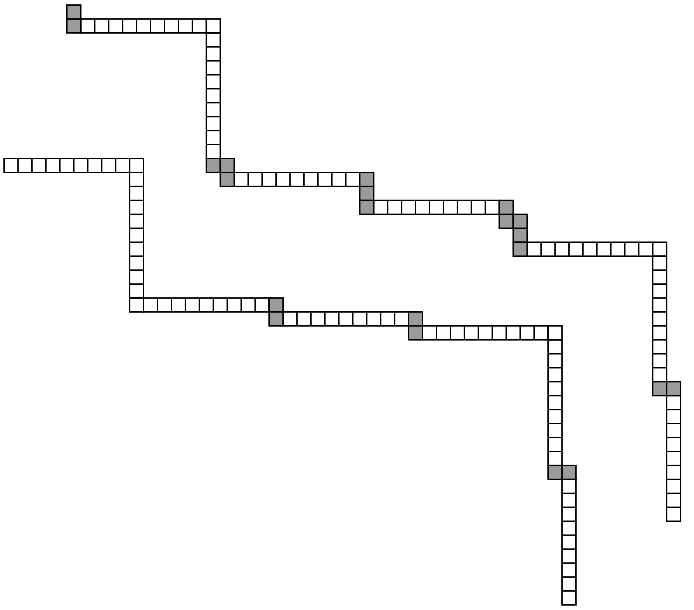
$$\mathsf{bw}(\lambda_{2n}) = \underbrace{+-\dots+-}_{2n}.$$

For that we will use the oriented paintbox construction from [7], see Definition 5.4 and the paragraph above Proposition 6.3 in that paper. Following its notation, it remains to prove that the probability $\mathbb{P}(\Pi_{2n+1} = \pi_{2n+1})$ is non-zero, where in one-line notation the permutation $\pi_{2n+1} \in S_{2n+1}$ is given by

$$\pi_{2n+1} = 1, 2n+1, 2, 2n, 3, 2n-1, \dots n-1, n+3, n, n+2, n+1.$$

This fact immediately follows from the next observation. If $w \in W \setminus W_0$ contains infinitely many intervals, then we can place random points inside different intervals in the desired order. But if $w \in W \setminus W_0$ consists of finitely many intervals, then their common length is strictly less then 1 and we can place our random points inside that complement to w, which length is non-zero. $\square$

Remark 5.4.5. If $w \in W_0$, then the template t_w does not contain finite clusters except those one-symbol clusters which are not outermost and whose two neighbors are infinite clusters of the same sign. Such templates will be called *finite*. A template which is not finite will be called *semifinite*, see Figure 5.4.



$a)$ $b)$

Figure. 5.4: a) $\mathsf{z}(t)$ for the finite template $t = \overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$.
b) $\mathsf{z}(t)$ for the semifinite template $t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{1}{-}\overset{\infty}{+}\overset{2}{-}\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{2}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$.

Remark 5.4.6. Proposition 5.4.4 allows us to think of a finitary oriented paintbox $w = (w_+, w_-)$ as the infinite zigzag $\mathsf{z}(t_w)$ endowed with a tuple of real numbers. Namely, we attach lengths of the intervals from w_+ and w_- to infinite rows and columns of $\mathsf{z}(t_w)$ respectively. Moreover, we may identify the infinite zigzag $\mathsf{z}(t_w)$ with an infinite path τ which completely "fills" this zigzag, see the paragraph above Proposition 5.3.2. Starting from some point, this path τ looks like a collection of growing rows and columns, hence we can treat the lengths of intervals from w as frequencies of appearing new boxes in that rows and columns which grow infinitely large.

5.4.2 A useful lemma

Now we would like to discuss a lemma, which we will use to prove a semifinite analog of the Vershik-Kerov ring theorem for indecomposable semifinite harmonic functions on the zigzag graph, Theorem 5.8.1.

Let u be an m-tuple of adjacent oriented intervals; their lengths will be denoted by $u_1, \dots, u_m$. The only thing that differs u from a finitary oriented paintbox is that we do not impose any restrictions on lengths of the intervals. For any zigzag λ the expression $F_\lambda(u)$ is defined by the formula from Proposition 5.4.2. Equivalently, we can define this expression by Kerov's construction (5.2). Then it is obvious that $F_\lambda \mapsto F_\lambda(u)$ is a homomorphism of algebras $QSym \to \mathbb{R}$. We can also define a template t_u in the same manner as for finitary oriented paintboxes, see the paragraph above Proposition 5.4.4.

Now suppose that $\lambda \in \mathbb{Z}(t_u)$ and $\mathsf{bw}(\lambda)$ contains as many blocks as possible. Then each block of $\mathsf{bw}(\lambda)$ either corresponds to an interval of u or it is a one-symbol block that is placed between two blocks corresponding to equally oriented intervals. The blocks corresponding to intervals of u will be denoted by $\Lambda_1, \dots, \Lambda_m$. Recall that $|\Lambda_i|$ denotes the number of symbols in the binary word Λ_i.

Let us introduce the following notation $n_1(\lambda) = |\Lambda_1|$, $n_m(\lambda) = |\Lambda_m|$ and for any $i = 2, \dots, m-1$ we set

$$n_i(\lambda) = \begin{cases} |\Lambda_i| + 1, & \text{if the } i-1\text{-st}, i\text{-th}, i+1\text{-st intervals of } u \text{ are equally oriented,} \\ |\Lambda_i|, & \text{if the } i\text{-th interval of } u \text{ has exactly one neighbor of the same orientation,} \\ |\Lambda_i| - 1 & \text{otherwise.} \end{cases}$$

Let us introduce more notation: $s(u) = |S(u)|$, where

$$S(u) = \{i \mid 1 \le i \le m\colon \text{the } i\text{-th and the } i+1\text{-st intervals of } u \text{ have different orientations}\}.$$

Lemma 5.4.7. *Assume that $\lambda \in \mathcal{Z}(t_u)$ and $\mathsf{bw}(\lambda)$ contains as many blocks as possible. Then*

$$F_\lambda(u) = u_1^{n_1(\lambda)} \dots u_m^{n_m(\lambda)} \cdot \sum_{\rho \in \{0,1\}^{s(u)}} \prod_{i \in S(u)} u_{i+\rho(i)},$$

where the sum is taken over all $s(u)$-tuples ρ, consisting of 0's and 1's.

Proof. The claim follows from the very definition of $F_\lambda(u)$. Namely, the sum corresponds to all possible splittings of λ mentioned above Proposition 5.4.2. $\square$

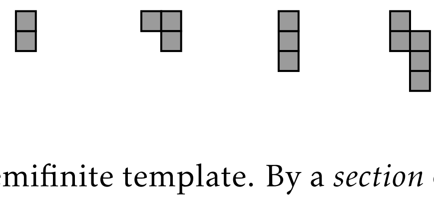
5.5 Semifinite templates

From Proposition 3.4.6 and Observation 5.3.3 with $k = 1$ it follows that for a finite template t the graph $\mathcal{Z}(t)$ possess no strictly positive indecomposable semifinite harmonic functions, hence $\mathcal{Z}(t)$ can not be realised as the support of an indecomposable semifinite harmonic function on the zigzag graph. Thus, below we will be interested only in semifinite templates.

It turns out that for any semifinite template t the coideal $\mathcal{Z}(t)$ can be realised as the support of an indecomposable semifinite harmonic function on the zigzag graph. Moreover, for indecomposable semifinite harmonic functions with the common support $\mathcal{Z}(t)$ the finiteness ideal depends only on t. In the present section we describe this finiteness ideal. Some examples are given in the next section.

Definition 5.5.1. Let t be a semifinite template. By a *separating cluster* of t we mean a one-symbol cluster which is not an outermost cluster of t and whose two neighbors are infinite clusters of the same sign. By the *zigzag flange* of t we call a tuple of binary words each of which consists of finite but not separating clusters of t standing nearby. The zigzag flange will be denoted by $\mathsf{fl}(t)$.

For instance, if we take $t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{1}{-}\overset{\infty}{+}\overset{2}{-}\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{2}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$, which is the semifinite template from Figure 5.4, then $\mathsf{fl}(t) = (\overset{1}{-}, \overset{1}{+}\overset{1}{-}, \overset{2}{-}, \overset{1}{-}\overset{1}{+}\overset{2}{-})$. So, the words from this zigzag flange correspond to the first, second, third, and fourth grey zigzags on the Figure 5.4b):



Definition 5.5.2. Let t be a semifinite template. By a *section* of t we mean a maximal collection of consecutive clusters that form a finite template.

Note that the words from the zigzag flange of t split t into sections.

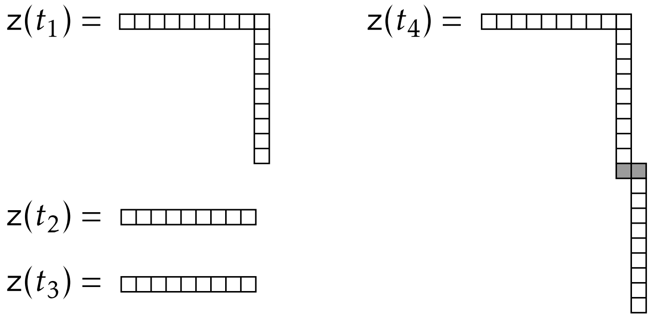
For the above t the sections are

$$t_1 = \overset{\infty}{+}\overset{\infty}{-},\ t_2 = \overset{\infty}{+},\ t_3 = \overset{\infty}{+},\ t_4 = \overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$$

and the splitting of t into sections is given by

$$\Big(\overset{1}{-},\ t_1,\ \overset{1}{+}\overset{1}{-},\ t_2,\ \overset{2}{-},\ t_3,\ \overset{1}{-}\overset{1}{+}\overset{2}{-},\ t_4\Big).$$

In terms of infinite zigzags, we split $\mathsf{z}(t)$ into infinite zigzags corresponding to sections of t. For example, $\mathsf{z}(t)$ from Figure 5.4b) is split by the first, second, third, and fourth grey zigzags into



Remark 5.5.3. There are analogs of sections and zigzag flanges for the saturated primitive coideals of the Young graph, see the picture on page 148 in [31]. Namely, each saturated primitive coideal of the Young graph looks like a thick infinite hook with a flange consisting of a single Young diagram. In our case of the zigzag graph, sections with a zigzag flange play the role of that infinite hook with a Young diagram.

Definition 5.5.4. Let us set $J(t) = \bigcup_r \mathbb{Z}(r)$, where the union is taken over all r obtained from t by removing a single symbol from some cluster corresponding to a block of a binary word from the zigzag flange $\mathsf{fl}(t)$.

Note that r from the definition above may fail to be a template, since r can contain two neighbor clusters of the same sign. Anyway, the construction of $\mathbb{Z}(r)$ remains unchanged. Namely, $\mathbb{Z}(r)$ is the coideal of $\mathbb{Z}$ corresponding to any path passing through the zigzags corresponding to binary words obtained from r by replacing infinite clusters with long enough blocks. This means that we merge two neighbor clusters of the same sign in r into a bigger cluster by adding their lengths.

The ideal $\mathbb{Z}(t)\backslash J(t)$ of $\mathbb{Z}(t)$ is going to be the finiteness ideal of any strictly positive indecomposable semifinite harmonic function on $\mathbb{Z}(t)$. Recall that these functions are in an obvious bijection with indecomposable semifinite harmonic functions on $\mathbb{Z}$ whose support equals $\mathbb{Z}(t)$. Now we would like to describe the ideal $\mathbb{Z}(t)\backslash J(t)$ in more details.

Let t be a semifinite template with k sections $t_1,\dots,t_k$. Assume that $\mathsf{fl}(t) = (a_0,\dots,a_k)$ and the splitting of t into sections looks like

$$t = (a_0,\ t_1,\ a_1,\ \dots\ ,a_{k-1},\ t_k,\ a_k).$$

If a_0 or a_k is the empty binary word, then we should merely ignore it in all what follows.

For binary words a and b we write $a > b$ if and only if the number of symbols in a is greater then the number of symbols in b and b can be obtained from a by removing some symbols. The symbol $\sqcup$ denotes the concatenation of binary words. For instance, $- \sqcup + = -+$, $\overset{2}{-} \sqcup \overset{3}{-} = \overset{5}{-}$ and the empty binary word is the identity for $\sqcup$.

Lemma 5.5.5.

1) If $\lambda \in \mathcal{Z}(t)\backslash J(t)$, then

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k$$

for some $\lambda^{(i)} \in \mathcal{Z}(t_i)$, which are uniquely defined.

2) The following map

$$\mathcal{Z}(t)\backslash J(t) \to \mathcal{Z}(t_1) \times \ldots \times \mathcal{Z}(t_k), \quad \lambda \mapsto (\lambda^{(1)}, \ldots, \lambda^{(k)}),$$

provided by the first part of the lemma, defines an embedding of graded graphs[2], see Definition 4.2.1. Moreover, the image of this embedding is an ideal.

Proof. 1) Obviously, we can write

$$\mathsf{bw}(\mathcal{Z}(t)) = \left\{ b_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup b_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup b_k \;\middle|\; b_i \leq a_i, \lambda^{(i)} \in \mathcal{Z}(t_i) \right\},$$
$$\mathsf{bw}(J(t)) = \left\{ b_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup b_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup b_k \;\middle|\; b_i \leq a_i \text{ and } \exists j \colon b_j < a_j;\ \lambda^{(i)} \in \mathcal{Z}(t_i) \right\}.$$

So, it suffices to show that if $\lambda \in \mathcal{Z}(t)\backslash J(t)$ and

$$\begin{aligned} \mathsf{bw}(\lambda) &= a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k = \\ &= a_0 \sqcup \mathsf{bw}(\widetilde{\lambda^{(1)}}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\widetilde{\lambda^{(k)}}) \sqcup a_k, \end{aligned} \tag{5.3}$$

for some $\lambda^{(i)}, \widetilde{\lambda^{(i)}} \in \mathcal{Z}(t_i)$, then $\widetilde{\lambda^{(i)}} = \lambda^{(i)}$.

Let us denote by m the natural number such that $\lambda^{(1)} = \widetilde{\lambda^{(1)}}, \ldots, \lambda^{(m-1)} = \widetilde{\lambda^{(m-1)}}$, but $\lambda^{(m)} \neq \widetilde{\lambda^{(m)}}$. Equation (5.3) yields $|\mathsf{bw}(\lambda^{(m)})| \neq |\mathsf{bw}(\widetilde{\lambda^{(m)}})|$, hence we may assume that $|\mathsf{bw}(\widetilde{\lambda^{(m)}})| > |\mathsf{bw}(\lambda^{(m)})|$. Then we can write

$$\mathsf{bw}(\widetilde{\lambda^{(m)}}) = \mathsf{bw}(\lambda^{(m)}) \sqcup \delta \in \mathsf{bw}(\mathcal{Z}(t_m))$$

for some non-empty binary word δ. From equation (5.3) it follows that the first symbol in δ is the same as in a_m. Thus, the condition $\mathsf{bw}(\lambda^{(m)}) \sqcup \delta \in \mathsf{bw}(\mathcal{Z}(t_m))$ implies that $\lambda \in J(t)$. This contradiction proves the first part of the lemma.

2) Let us denote the map $\lambda \mapsto (\lambda^{(1)}, \ldots, \lambda^{(k)})$ by f. Let us show that $\lambda, \mu \in \mathcal{Z}(t)\backslash J(t)$ are joined by an edge if and only if $f(\lambda)$ and $f(\mu)$ are joined by an edge. Obviously, if $f(\lambda) \nearrow f(\mu)$, then $\lambda \nearrow \mu$. Suppose now that $\lambda \nearrow \mu$. From the first part of the lemma it follows that

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda^{(k)}) \sqcup a_k \in \mathsf{bw}(\mathcal{Z}(t)\backslash J(t))$$

and

$$\mathsf{bw}(\mu) = a_0 \sqcup \mathsf{bw}(\mu^{(1)}) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\mu^{(k)}) \sqcup a_k \in \mathsf{bw}(\mathcal{Z}(t)\backslash J(t))$$

for some $\lambda^{(i)}, \mu^{(i)} \in \mathcal{Z}(t_i)$.

The condition $\lambda \nearrow \mu$ means that we can obtain $\mathsf{bw}(\lambda)$ by removing a symbol from $\mathsf{bw}(\mu)$. Therefore, this symbol can be deleted only from some $\mathsf{bw}(\mu^{(i)})$. The thing is that we can not remove that symbol from some a_i, because then the result belongs to $\mathsf{bw}(J(t))$, but $\lambda \notin J(t)$. Thus, $f(\lambda)$ and $f(\mu)$ are joined by an edge.

Suppose that $\lambda \in \mathcal{Z}(t)\backslash J(t)$ and $(\mu_1, \ldots, \mu_k) \in \mathcal{Z}(t_1) \times \ldots \times \mathcal{Z}(t_k)$. It is straightforward to check that $f(\lambda) \nearrow (\mu_1, \ldots, \mu_k)$ yields $(\mu_1, \ldots, \mu_k) \in \mathrm{Im}(f)$. Thus, $\mathrm{Im}(f)$ is an ideal.

□

[2]By an embedding $f \colon \Gamma_1 \to \Gamma_2$ of graded graphs we mean an injective map between the sets of vertices such that for any $\lambda, \mu \in \Gamma_1$ we have $\lambda \nearrow \mu$ if and only if $f(\lambda) \nearrow f(\mu)$.

The map provided by Lemma 5.5.5 is not surjective by a trivial reason, since the element

$$(\varnothing,\ldots,\varnothing)\in \mathcal{Z}(t_1)\times\ldots\times\mathcal{Z}(t_k),$$

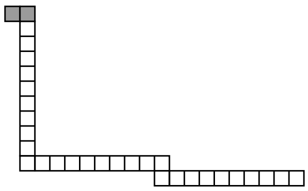
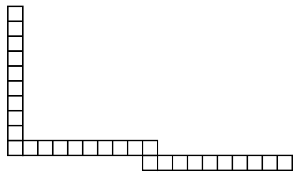
corresponding to the empty zigzag at each factor, can not belong to the image. In fact, this is not the only obstacle for this map to be surjective; see examples below.

5.6 Examples

For a binary word a we use the following notation

$$\mathcal{Z}(t)^a=\{\lambda\in\mathcal{Z}(t)\mid \mathsf{bw}(\lambda)\geq a\}.$$

Example 5.6.1. Take $t=\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}$, then its zigzag flange consists of the binary word $a_0=\overset{1}{+}$ and the single section is $t_1=\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}$.

$\mathsf{z}(t)=$  $\mathsf{z}(t_1)=$ 

Next, $J(t)=\mathcal{Z}(t_1)$. It is obvious that $J(t)$ and $\mathcal{Z}(t)^{+--}$ do not intersect. Moreover, one can check that $\mathcal{Z}(t)=J(t)\cup\mathcal{Z}(t)^{+--}$. In an expanded form it reads as

$$\mathcal{Z}\left(\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right)=\mathcal{Z}\left(\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right)\cup\mathcal{Z}\left(\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right)^{+--}.$$

Let us prove this. Suppose that $\lambda\in\mathcal{Z}(t)$. We have to consider several cases.

- $\mathsf{bw}(\lambda)$ has 5 blocks. Then $\lambda\in\mathcal{Z}(t)^{+--}$.
- $\mathsf{bw}(\lambda)$ has 4 blocks. Since there are only two types of binary words consisting of 4 blocks, it follows that either $\mathsf{bw}(\lambda)=\overset{n_1}{+}\overset{n_2}{-}\overset{n_3}{+}\overset{n_4}{-}$ or $\mathsf{bw}(\lambda)=\overset{n_1}{-}\overset{n_2}{+}\overset{n_3}{-}\overset{n_4}{+}$ for some strictly positive integers n_1,n_2,n_3,n_4. If the former, then $\lambda\in\mathcal{Z}(t)^{+--}$; if the latter, then $n_3=1$ and $\lambda\in\mathcal{Z}(t_1)$.
- $\mathsf{bw}(\lambda)$ has 3 blocks. Then for some strictly positive integers n_1,n_2,n_3 one of the following holds:
  - $\mathsf{bw}(\lambda)=\overset{n_1}{+}\overset{n_2}{-}\overset{n_3}{+}$ with $n_2\geq 2$ and then $\lambda\in\mathcal{Z}(t)^{+--}$;
  - $\mathsf{bw}(\lambda)=\overset{n_1}{+}\overset{1}{-}\overset{n_3}{+}$ and $\lambda\in\mathcal{Z}(t_1)$;
  - $\mathsf{bw}(\lambda)=\overset{n_1}{-}\overset{n_2}{+}\overset{n_3}{-}$ with $n_3\geq 2$ and then $\lambda\in\mathcal{Z}(t)^{+--}$;
  - $\mathsf{bw}(\lambda)=\overset{n_1}{-}\overset{n_2}{+}\overset{1}{-}$ and $\lambda\in\mathcal{Z}(t_1)$.
- $\mathsf{bw}(\lambda)$ has 2 blocks. Then for some strictly positive integers n_1,n_2 one of the following holds:
  - $\mathsf{bw}(\lambda)=\overset{n_1}{-}\overset{n_2}{+}$ and $\lambda\in\mathcal{Z}(t_1)$;
  - $\mathsf{bw}(\lambda)=\overset{n_1}{+}\overset{n_2}{-}$ with $n_2\geq 2$ and then $\lambda\in\mathcal{Z}(t)^{+--}$;

– $\mathsf{bw}(\lambda) = \overset{n_1}{+}\overset{1}{-}$ and $\lambda \in \mathcal{Z}(t_1)$;

- $\mathsf{bw}(\lambda)$ consists of a single block. Then $\lambda \in \mathcal{Z}(t_1)$.

Next, we describe the the map provided by Lemma 5.5.5.

For any zigzag $\lambda \in \mathcal{Z}(t)^{+--}$ we can write $\mathsf{bw}(\lambda) = \overset{1}{+} \sqcup \mathsf{bw}(\overline{\lambda})$ for a unique $\overline{\lambda} \in \mathcal{Z}(t_1)$ such that $\mathsf{bw}(\overline{\lambda})$ contains at least two minuses. Thus, the map provided by Lemma 5.5.5 is given by

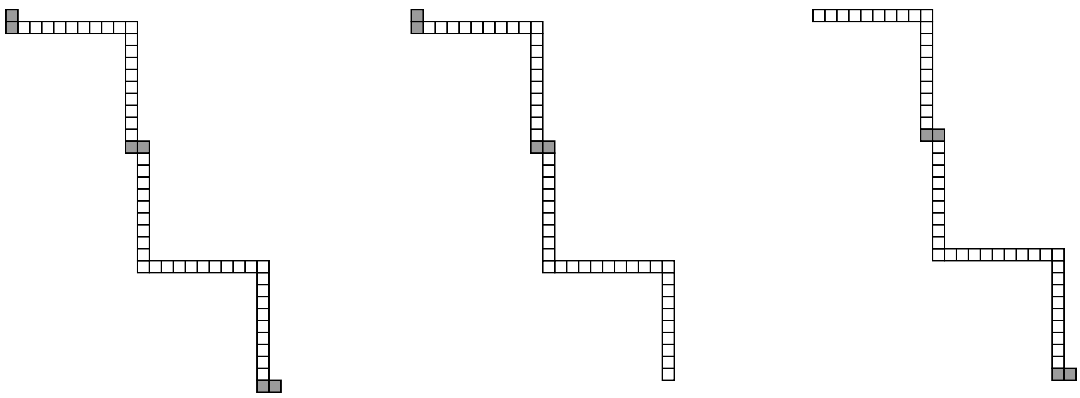
$$\mathcal{Z}\left(\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right)^{+--} \longrightarrow \mathcal{Z}\left(\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right), \quad \lambda \mapsto \overline{\lambda}.$$

This map is not surjective, since $\mathsf{bw}(\overline{\lambda})$ contains at least two minuses. The image of this map is the ideal generated by $\overset{2}{-}$, that is $\mathcal{Z}\left(\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\right)^{--}$.

The ideal $\mathcal{Z}(t)\backslash J(t)$ in the previous example is generated by a single zigzag. The next example shows that this is not the case in general.

Example 5.6.2. Take $t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}$, then $a_0 = \overset{1}{-}$, $a_1 = \overset{1}{+}$, $t_1 = \overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}$, and

$$J(t) = \mathcal{Z}\left(\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\right) \cup \mathcal{Z}\left(\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\right).$$



$a)$ $b)$ $c)$

Figure. 5.5: $a)$ $\mathsf{z}(t)$, $b)$ $\mathsf{z}\left(\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\right)$, $c)$ $\mathsf{z}\left(\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\right)$.

It is not difficult to check that

$$\mathcal{Z}(t)\backslash J(t) = \mathcal{Z}(t)^{-+-+-+-+} \cup \mathcal{Z}(t)^{-\overset{2}{+}-\overset{2}{+}-+}.$$

Note that the first summand, which is isomorphic to the 5-th dimensional Pascal graph $\mathbb{P}_5$, corresponds to the binary words having the maximal possible number of blocks.

We will show that the ideal $\mathcal{Z}(t)\backslash J(t)$ can not be generated by a single zigzag.

Let us denote by $\downarrow(\lambda)$ the set of all lower adjacents of λ. Then

$$\begin{aligned}\downarrow\left(-\overset{2}{+}-\overset{2}{+}-+\right) = \Big\{&\overset{2}{+}-\overset{2}{+}-+, -+-\overset{2}{+}-+, -\overset{4}{+}-+,\\ &-\overset{2}{+}-+-+, -\overset{2}{+}-\overset{3}{+}, -\overset{2}{+}-\overset{2}{+}-\Big\}\end{aligned}$$

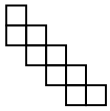
and

$$\begin{aligned}\downarrow(-+-+-+-+) = \Big\{&+-+-+-+, \overset{2}{-}+-+-+, -\overset{2}{+}-+-+, -+\overset{2}{-}+-+,\\ &-+-\overset{2}{+}-+, -+-+\overset{2}{-}+, -+-+-\overset{2}{+}, -+-+-+-\Big\}.\end{aligned}$$

Thus,

$$\downarrow(-+-+-+-+)\cap\downarrow(-\overset{2}{+}-\overset{2}{+}-+)=\left\{-\overset{2}{+}-+-+,-+-\overset{2}{+}-+\right\}\subset J(t)$$

and the ideal $\mathcal{Z}(t)\backslash J(t)$ can not be generated by a single zigzag.

$\lambda_1 =$  $\mathsf{bw}(\lambda_1) = -+-+-+-+$

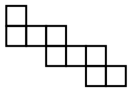
$\lambda_2 =$  $\mathsf{bw}(\lambda_2) = -\overset{2}{+}-\overset{2}{+}-+$

Figure. 5.6: Generators of the ideal $\mathcal{Z}(t)\backslash J(t)$ for $t=\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}$.

Thus, the graph $\mathcal{Z}(t)\backslash J(t)$ is is not a branching graph but a graded graph, see Figure 5.7.

Let us describe the map from Lemma 5.5.5.

It is easy to check that the first block of $\mathsf{bw}(\lambda)$ for $\lambda\in\mathcal{Z}(t)\backslash J(t)$ must be the negative one-symbol block and the last block of $\mathsf{bw}(\lambda)$ must be the positive one symbol block. So, the desired map is given by

$$\mathcal{Z}(t)^{-+-+-+-+}\cup\mathcal{Z}(t)^{-\overset{2}{+}-\overset{2}{+}-+}\longrightarrow\mathcal{Z}\left(\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\right),\quad\lambda\mapsto\overline{\lambda},$$

where $t=\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}$ and $\mathsf{bw}(\lambda)=-\sqcup\mathsf{bw}(\overline{\lambda})\sqcup+$.

In the next example we propose a sufficient condition for $\mathcal{Z}(t)\backslash J(t)$ to be generated by a single zigzag. This condition is not necessary, because the template t from Example 5.6.1 does not satisfy it, however the corresponding ideal is generated by a single zigzag.

Example 5.6.3. Let t be a semifinite template. We say that a cluster of t is *internal* if t neither begins nor ends with this cluster. Let us denote by a_t the binary word obtained from t by applying the following rules:

- if t begins or ends with an infinite cluster, then this infinite cluster is removed;
- each internal infinite cluster of t having an infinite neighbour is replaced by a one-symbol cluster of the same sign;
- each infinite cluster standing between two finite clusters is removed.

For instance, $a_t=-\overset{2}{+}-+\overset{4}{-}$ for $t=\overset{\infty}{+}\overset{\infty}{-}\overset{2}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\overset{3}{-}$.

It is not difficult to check that $\mathcal{Z}(t)^{a_t}$ is isomorphic as a graded graph to the Pascal graph of an appropriate dimension.

Suppose that t satisfies the constraints:

- t avoids the following patterns

$$\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\quad\text{and}\quad\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-};$$

- t does not begin with

$$\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\quad\text{or}\quad\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+};$$

- t does not end with

$$\overset{\infty}{-}\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}\quad\text{or}\quad\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}.$$

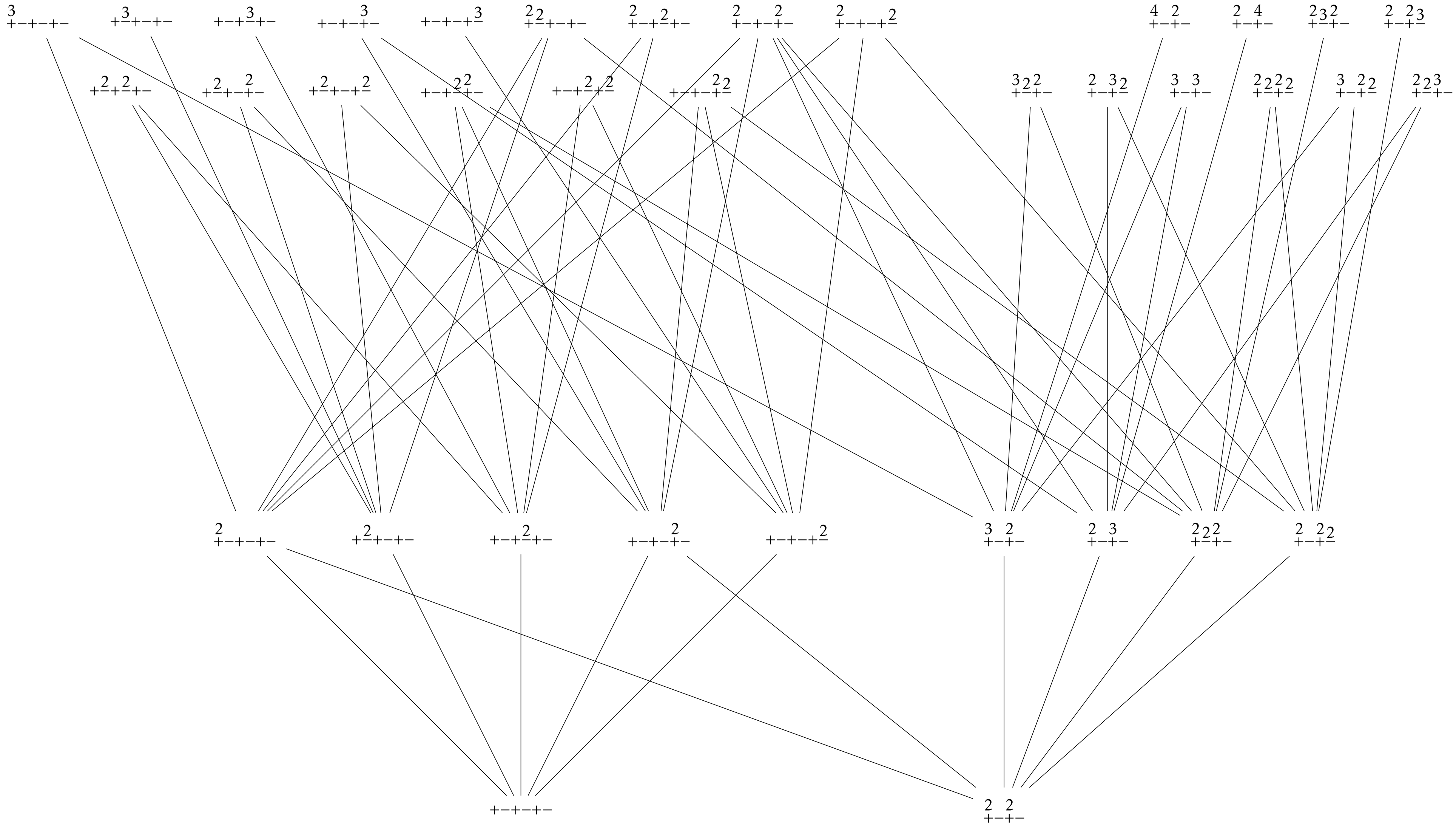


Figure. 5.7: The first three levels of the image of the map $\mathcal{Z}(t)\backslash J(t) \hookrightarrow \mathcal{Z}\left(\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\right)$ provided by Lemma 5.5.5 for $t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}$, see Example 5.6.2. Zigzags are represented by the corresponding binary words.

Then

$$\mathcal{Z}(t)\backslash J(t) = \mathcal{Z}(t)^{a_t}.$$

Let us prove this. We will analyze the set $\mathcal{Z}(t)\backslash\mathcal{Z}(t)^{a_t}$ in order to prove that it equals $J(t)$. The proof splits into the following parts:

1) if r is obtained from t by removing a separating symbol and merging the two infinite clusters standing near that separating symbol, then there exists r' that can be obtained from t by removing a symbol from a cluster corresponding to a word from the zigzag flange of t and such that $\mathcal{Z}(r) \subset \mathcal{Z}(r')$;

2) $J(t) \subset \mathcal{Z}(t)\backslash\mathcal{Z}(t)^{a_t}$;

3) $J(t) \supset \mathcal{Z}(t)\backslash\mathcal{Z}(t)^{a_t}$

The first part of the proof. Since $\mathcal{Z}(r')$ is a coideal of $\mathcal{Z}(t)$, it is sufficient to show that any binary word $a \in \mathsf{bw}(\mathcal{Z}(r))$ which has as many blocks as possible belongs to $\mathsf{bw}(\mathcal{Z}(r'))$. Moreover, we can assume that the blocks of a are large enough that is, the blocks corresponding to finite clusters are of maximal lengths and the blocks corresponding to infinite clusters are of length, say, N, where N is large enough.

We have to deal with the one-symbol cluster of t which is a part of one of the following patterns $\overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}$ or $\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$. Let us restrict ourselves only to the first case. Then we can rewrite t as

$$t = (p_1, \overset{\infty}{+}\overset{1}{-}\overset{\infty}{+}, p_2),$$

where p_1 and p_2 are such that either p_1 ends with a finite cluster or p_2 starts with it. Note that here we used the constraints on t listed above. The template r looks like

$$r = (p_1, \overset{\infty}{+}, p_2)$$

and the induced splitting of a reads as

$$a = a^{(1)} \sqcup \overset{N}{+} \sqcup a^{(2)},$$

where $a^{(1)} \in \mathsf{bw}(\mathcal{Z}(p_1))$ and $a^{(2)} \in \mathsf{bw}(\mathcal{Z}(p_2))$ contain as many blocks as possible. Then we can obtain the desired template r' by removing a symbol from that finite cluster of t with which p_1 ends or p_2 begins.

The second part of the proof. Suppose that $a \in \mathsf{bw}(\mathcal{Z}(r))$ is a binary word consisting of maximal possible number of blocks which are large enough, where r is obtained from t by removing a symbol from a cluster corresponding to a word from the zigzag flange of t. We will assume that the blocks of a corresponding to infinite clusters of t are of the same length, which we denote by N. Then the number of infinite clusters in r is the same as in t and equals the number of blocks of length N in a. These blocks of length N split a into parts almost all of which are binary words from the zigzag flange of t, except one part, which differs from a binary word from $\mathsf{fl}(t)$ by a single symbol. We may write this splitting and the induced splitting of a_t as

$$a = a^{(1)} \sqcup \beta \sqcup a^{(2)}$$
$$a_t = a_t^{(1)} \sqcup \alpha \sqcup a_t^{(2)},$$

where α is a binary word from $\mathsf{fl}(t)$ and $\beta \nearrow \alpha$. Then one can easily see that $a^{(i)} \geq a_t^{(i)}$ for $i = 1, 2$, but

$$a^{(1)} \not\geq a_t^{(1)} \sqcup \delta_1 \quad \text{and} \quad a^{(2)} \not\geq \delta_2 \sqcup a_t^{(2)},$$

where δ_1 and δ_2 denote single symbols from the first and the last clusters of α.

Thus, $a \ngtrsim a_t$ and the claim follows.

The third part of the proof. For notational simplicity, let us assume that the first cluster of t is of sign plus and the total number of clusters in t is even. We denote this number by n. Then we can write

$$t = \overset{k_1}{+}\overset{k_2}{-}\dots\overset{k_{n-1}}{+}\overset{k_n}{-},$$

where $k_1, \dots, k_n$ is the tuple of formal multiplicities of t, some of which may be infinite.

Furthermore,

$$\mathsf{bw}(\mathcal{Z}(t)) = \{\overset{l_1}{+}\overset{l_2}{-}\dots\overset{l_{n-1}}{+}\overset{l_n}{-} \mid l_i \le k_i\}.$$

Let us denote by $I \subset \{1, 2, \dots, n\}$ the positions of finite clusters of t. Then

$$\mathsf{bw}\left(\mathcal{Z}(t)^{a_t}\right) = \left\{ \overset{l_1}{+}\overset{l_2}{-}\dots\overset{l_{n-1}}{+}\overset{l_n}{-} \;\middle|\; \begin{array}{l} \bullet\ l_i = k_i \text{ if } i \in I; \\ \bullet\ l_i \ge 1 \text{ if } i \notin I,\ i \ne 1, n, \text{ and } i+1 \notin I \text{ or } i-1 \notin I \end{array} \right\}.$$

So, if $\overset{l_1}{+}\overset{l_2}{-}\dots\overset{l_{n-1}}{+}\overset{l_n}{-} \in \mathsf{bw}(\mathcal{Z}(t) \setminus \mathcal{Z}(t)^{a_t})$, then at least one of the following conditions must hold

1) $l_i \le k_i - 1$ for some $i \in I$

2) $l_i = 0$ for some i such that $2 \le i \le n-1$, $i \notin I$, and $i+1 \notin I$ or $i-1 \notin I$.

Thus, from the first part of the proof it follows that to prove the desired inclusion it suffices to show that any binary word $a = \overset{l_1}{+}\overset{l_2}{-}\dots\overset{l_{n-1}}{+}\overset{l_n}{-}$ satisfying the second condition above belongs to $\mathsf{bw}(\mathcal{Z}(r))$ for some r which can be obtained from t by removing a symbol from a finite cluster. In order to do so, we rewrite t in the following way

$$t = s_1 \sqcup s \sqcup s_2$$

with s being the maximal template consisting only of infinite clusters and containing the i-th infinite cluster of t, where i is the number from the second condition above for our binary word a. Note that then s_2 either begins with a finite cluster or is empty; if the latter, then s_1 ends with a finite cluster. Anyway, it is obvious that we can obtain the desired r by removing a single symbol from a cluster of t which is neighbour to s and is an outermost cluster of s_1 or s_2.

5.7 Main result: harmonic functions on $\mathcal{Z}(t)$

From Proposition 3.4.6 and Observation 5.3.3 with $k = 1$ it follows that for a finite template t the graph $\mathcal{Z}(t)$ possess no strictly positive indecomposable semifinite harmonic functions, hence $\mathcal{Z}(t)$ can not be realised as the support of an indecomposable semifinite harmonic function on the zigzag graph. Then Propositions 3.3.16 and 5.3.2 imply that in order to describe all indecomposable semifinite harmonic functions on the zigzag graph it is sufficient to describe all strictly positive indecomposable semifinite harmonic functions on $\mathcal{Z}(t)$ for any semifinite template t. That is what we do in the present section.

Now we would like to introduce some strictly positive functions on the graph $\mathcal{Z}(t)$, see Definition 5.7.3. Below we prove that they are pairwise distinct and form an exhaustive list of indecomposable semifinite harmonic functions on $\mathcal{Z}(t)$.

Definition 5.7.1. By a *semifinite zigzag growth model* we call a pair (t, w), where t is a semifinite template having m infinite clusters and $w = (w_1, \dots, w_m)$ is an m-tuple of positive real numbers such that $w_1 + \dots + w_m = 1$.

Remark 5.7.2. We can assume that these real numbers $w_1, \dots, w_m$ are assigned to infinite clusters of t. Then we can identify a semifinite zigzag growth model (t, w) with the infinite zigzag $\mathrm{z}(t)$ endowed with a tuple of frequencies, see Remark 5.4.6. Furthermore, we can treat this w as a finitary oriented paintbox the i-th interval component of which is of length w_i; orientation of this interval component is defined by the sign of the corresponding infinite cluster of t: the orientation is positive if the cluster is positive and the orientation is negative if the cluster is negative.

Let t have k sections $t_1, \dots, t_k$. Assume that $\mathsf{fl}(t) = (a_0, \dots, a_k)$ and the splitting of t into sections looks like

$$t = (a_0,\ t_1,\ a_1,\ \dots\ , a_{k-1},\ t_k,\ a_k).$$

If a_0 or a_k is the empty binary word, then we should merely ignore it in all what follows.

Let (t, w) be a semifinite zigzag growth model. The splitting of t into sections gives us a splitting of w

$$w = v_1 \sqcup \ldots \sqcup v_k,$$

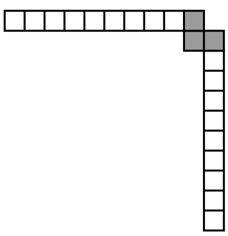
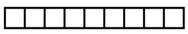
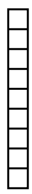
where each v_i is a tuple of real numbers from $w = (w_1, \dots, w_m)$ corresponding to the infinite clusters of t_i. Note that we may treat each v_i as a collection of oriented subintervals of $(0,1)$; the only thing that differs v_i from a finitary oriented paintbox is the total length of intervals from v_i, which may not be equal to 1.

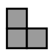
Definition 5.7.3. For any $\lambda \in \mathbb{Z}(t)$ we set

$$\varphi_{t,w}(\lambda) = \begin{cases} F_{\lambda^{(1)}}(v_1) \cdot \ldots \cdot F_{\lambda^{(k)}}(v_k), \text{ if } \lambda \in \mathbb{Z}(t) \backslash J(t) \\ +\infty, \text{ if } \lambda \in J(t), \end{cases}$$

where $\lambda \mapsto (\lambda^{(1)}, \dots, \lambda^{(k)})$ is the map provided by Lemma 5.5.5 and $F_{\lambda^{(i)}}(v_i)$ is defined by Kerov's construction (5.2) or by the formula from Proposition 5.4.2.

Example 5.7.4. Take $t = \overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{\infty}{-}$. Then a_0 and a_2 are empty binary words, $a_1 = \overset{1}{-}\overset{1}{+}$, $t_1 = \overset{\infty}{+}$, and $t_2 = \overset{\infty}{-}$.

$\mathrm{z}(t) =$  $\mathrm{z}(t_1) =$  $\mathrm{z}(t_2) =$ 

$a_1 =$ 

Next, $J(t) = \mathbb{Z}\left(\overset{\infty}{+}\overset{\infty}{-}\right)$ and

$$\mathbb{Z}(t) \backslash J(t) = \mathbb{Z}\left(\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{\infty}{-}\right)^{-+}.$$

Recall that the superscript denotes the zigzags which binary words contain $-+$.

The map

$$\mathbb{Z}\left(\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{\infty}{-}\right)^{-+} \longrightarrow \mathbb{Z}\left(\overset{\infty}{+}\right) \times \mathbb{Z}\left(\overset{\infty}{-}\right)$$

provided by Lemma 5.5.5 is given by $\overset{n}{+} - + \overset{m}{-} \mapsto (\overset{n}{+}, \overset{m}{-})$ and turns out to be as surjective as possible, see the paragraph below the proof of Lemma 5.5.5.

Let w_1 and w_2 be real positive numbers such that $w_1 + w_2 = 1$. Then

$$\varphi_{t,w}(\lambda) = \begin{cases} w_1^{n+1} w_2^{m+1}, \text{ if } \mathsf{bw}(\lambda) = \overset{n}{+} - + \overset{m}{-} \text{ with } n, m \geq 0, \\ \\ +\infty, \text{ if } \lambda \in \mathbb{Z}\left(\overset{\infty}{+}\overset{\infty}{-}\right). \end{cases}$$

It is straightforward to check that $\varphi_{t,w}$ is a harmonic function. Let us show that it is semifinite. This means that for any $\overset{n}{+}\overset{m}{-} \in \mathsf{bw}(J(t))$ we have to find an approximating sequence, see Remark 3.3.6. Note that we can assume that these n and m are large enough, since if $\lambda \geq \mu$ and $\{a_N\}_{N\geq 1}$ is an approximating sequence for λ, then $\{a_N\}_{N\geq 1}$ is an approximating sequence for μ as well. In fact, we will use only the bound $n, m \geq 2$. Below we treat binary word belonging to $\mathsf{bw}(\mathbb{Z}(t))$ as elements of $\mathrm{K}_0(\mathbb{Z}(t))$, see Section 3.

We argue that

$$a_N = N \cdot \overset{n-1}{+} - + \overset{m-1}{-}$$

for $N \geq 1$ form an approximating sequence for $\overset{n}{+}\overset{m}{-}$. Let us prove this.

Since

$$\varphi_{t,w}(a_N) = N \cdot w_1^n w_2^m$$

it follows that

$$\varphi_{t,w}(a_N) < +\infty$$

and

$$\varphi_{t,w}(a_N) \to +\infty \text{ as } N \to +\infty.$$

Thus, it suffices to show that $\overset{n}{+}\overset{m}{-} \geq_K a_N$; for the definition of $\geq_K$ see Section 3.

By the harmonicity condition for any $N \geq 1$ we can write

$$\overset{n}{+}\,\overset{m}{-} = \sum_{\substack{\lambda \in \mathbb{Z}(t): \\ |\lambda| = n+m+N+1}} \dim(\overset{n}{+}\overset{m}{-}, \lambda) \cdot \lambda \geq_K$$

$$\sum_{\substack{n_1, m_1 \geq 0: \\ n_1+m_1=N}} \dim\left(\overset{n}{+}\overset{m}{-}, \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}\right) \cdot \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}.$$

The harmonicity condition also implies

$$a_N = N \cdot \sum_{\substack{n_1, m_1 \geq 0: \\ n_1+m_1=N}} \dim\left(\overset{n-1}{+} - + \overset{m-1}{-}, \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}\right) \cdot \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}.$$

Note that

$$\dim\left(\overset{n-1}{+} - + \overset{m-1}{-}, \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}\right) = \binom{N}{n_1, m_1},$$

where $\binom{a+b}{a, b}$ denotes the binomial coefficient $\dfrac{(a+b)!}{a!b!}$.

Then the desired claim immediately follows from the next observation.

$$\dim\left(\overset{n}{+}\overset{m}{-}, \overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}\right) = n_1 \cdot \binom{N}{n_1, m_1} + m_1 \cdot \binom{N}{n_1, m_1} = N \cdot \binom{N}{n_1, m_1}. \tag{5.4}$$

Let us explain the origin of the first summand in (5.4). It equals the number of the following paths in the zigzag graph going from $\overset{n}{+}\overset{m}{-}$ to $\overset{n_1+n-1}{+} - + \overset{m_1+m-1}{-}$. Up to some point we increase only the two blocks of $\overset{n}{+}\overset{m}{-}$ and then we add the plus that increases the number of blocks in the word; after that, we increase only the outermost blocks again. So, we may treat each such path as an ordered collection of pluses and minuses, which we add to the original word $\overset{n}{+}\overset{m}{-}$; one plus, which increases the number of blocks, is marked. This collection is of length N and there are n_1 pluses in it. Only two things that may vary are positions of ordinary pluses, which we add to the leftmost block, and the position of that special plus that increases the number of blocks. So, we have to choose n_1 elements from the set of N elements and mark one of the chosen elements. The number of all such choices equals

$$n_1 \cdot \binom{N}{n_1, m_1}.$$

The second summand in (5.4) comes from the similar picture but for minuses. In fact, we have just proved the inequality $\geq$ in (5.4). But one can easily see that this is indeed an equality for $n, m \geq 2$.

Finally, we shall prove that $\varphi_{t,w}$ is indecomposable. Suppose that ψ is a finite or semifinite harmonic function on $\mathbb{Z}(t)$ such that $\varphi_{t,w} \geq \psi$. The ideal $\mathbb{Z}(t)\backslash J(t)$ is isomorphic to the 2-dimensional Pascal graph $\mathbb{P}_2$ and the restriction of $\varphi_{t,w}$ to this ideal is an indecomposable harmonic function. Thus, $\varphi_{t,w}$ and ψ are proportional on $\mathbb{Z}(t)\backslash J(t)$ as desired.

Theorem 5.7.5.

1) *For any semifinite zigzag growth model (t,w) the function $\varphi_{t,w}$ is a semifinite indecomposable harmonic function on $\mathbb{Z}(t)$.*

2) *Any strictly positive semifinite indecomposable harmonic function on the graph $\mathbb{Z}(t)$ is proportional to $\varphi_{t,w}$ for some semifinite zigzag growth model (t,w).*

3) *The functions $\varphi_{t,w}$ are distinct for distinct semifinite zigzag growth models (t,w).*

To prove this theorem we need the following lemma.

Lemma 5.7.6. *Let φ be a strictly positive harmonic function on $\mathbb{Z}(t)$. Then for any $\lambda \in J(t)$ we have $\varphi(\lambda) = +\infty$.*

Proof. Recall that

$$(J(t)) = \left\{ b_0 \sqcup (\lambda^{(1)}) \sqcup b_1 \sqcup \ldots \sqcup (\lambda^{(k)}) \sqcup b_k \;\middle|\; b_i \leq a_i \text{ and } \exists j\colon b_j < a_j;\ \lambda^{(i)} \in \mathbb{Z}(t_i) \right\}.$$

The set on which a harmonic function takes the value $+\infty$ is a coideal. Therefore, without loss of generality, we may assume that

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda_1) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda_j) \sqcup b_j \sqcup \mathsf{bw}(\lambda_{j+1}) \sqcup a_{j+1} \sqcup \ldots \sqcup \mathsf{bw}(\lambda_k) \sqcup a_k \tag{5.5}$$

for some j, where $b_j \nearrow a_j$ and each $\mathsf{bw}(\lambda_i) \in \mathsf{bw}(\mathbb{Z}(v_i))$ contains as many blocks as possible and all these blocks are large enough.

We will consider two cases:

1) the block of a_j from which we remove a symbol to obtain b_j is not an outermost block of a_j or it consists of 2 or more symbols;

2) we remove a symbol from an outermost block of a_j which is of length 1.

The first case. We denote by $\bar{t}$ the template obtained from t by replacing a_j with b_j. Then $\lambda \in \mathbb{Z}(\bar{t})$ and we can write

$$\left(\mathbb{Z}(t)\right)^{\lambda} = \left(\mathbb{Z}(\bar{t})\right)^{\lambda} \sqcup \left(\mathbb{Z}(t)\right)^{\lambda'}, \tag{5.6}$$

where λ' is defined as follows

$$\mathsf{bw}(\lambda') = a_0 \sqcup \mathsf{bw}(\lambda_1) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda_j) \sqcup a_j \sqcup \mathsf{bw}(\lambda_{j+1}) \sqcup a_{j+1} \sqcup \ldots \sqcup \mathsf{bw}(\lambda_k) \sqcup a_k.$$

Superscripts λ and λ' in (5.6) mean that we take all the zigzags that are greater than or equal to λ and λ' respectively.

Note that there is an isomorphism of graded graphs

$$\left(\mathbb{Z}(\bar{t})\right)^{\lambda} \to \left(\mathbb{Z}(t)\right)^{\lambda'},$$

which adds back the removed symbol $b_j \mapsto a_j$. This map is indeed an isomorphism, since each of these graphs is isomorphic to the Pascal graph of an appropriate dimension. Finally, from Lemma 3.5.6 it follows that $\varphi(\lambda) = +\infty$.

The second case. We may assume that $a_j = +-$ or $a_j = -+$. The point is that for all other a_j we can find a vertex which majorizes λ and satisfies the conditions of the first case above. We will restrict ourselves to the case $a_j = -+$. Then we may rewrite (5.5) as

$$\mathsf{bw}(\lambda) = a_0 \sqcup \mathsf{bw}(\lambda_1) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda_j) \sqcup \mathsf{bw}(\lambda_{j+1}) \sqcup \ldots \sqcup \mathsf{bw}(\lambda_k) \sqcup a_k$$

for some $\lambda_i \in \mathbb{Z}(t_i)$, because when we delete a symbol from $-+$ inside $\mathsf{bw}(\lambda)$ the remaining symbol is merged into $\mathsf{bw}(\lambda_j)$ or $\mathsf{bw}(\lambda_{j+1})$. Since each $\mathsf{bw}(\lambda_i)$ contains as many blocks as possible and their lengths are large enough, it follows that $\mathsf{bw}(\lambda_j) = \alpha \sqcup \overset{n}{+}$ for some "large enough" binary word α and a natural number n.

Let us define a zigzag λ' as follows

$$\begin{aligned} \mathsf{bw}(\lambda') = a_0 \sqcup \mathsf{bw}(\lambda_1) \sqcup a_1 \sqcup \ldots \sqcup \mathsf{bw}(\lambda_{j-1}) \sqcup a_{j-1} \\ \sqcup \alpha \sqcup \overset{n-1}{+} \sqcup \overset{1}{-} \sqcup \overset{1}{+} \sqcup \mathsf{bw}(\lambda_{j+1}) \sqcup a_{j+1} \sqcup \ldots \sqcup \mathsf{bw}(\lambda_{k-1}) \sqcup a_k. \end{aligned}$$

Now we are ready to apply Lemma 3.5.3. Below we use the notation from this lemma.

Let $I \subset \mathbb{Z}(t)$ be the ideal corresponding to the binary words that contain as many blocks as possible, provided that all blocks corresponding to the zigzag flange of t are of maximal lengths. Obviously, this ideal I is isomorphic as a graded graph to the Pascal graph of an appropriate dimension and $\lambda' \in I$, since α, which appeared in the definition of λ', is "large enough". Then $\dim(\lambda', \eta)$ on the right hand side of inequality (3.7) from Lemma 3.5.3 is a multinomial coefficient which arguments are merely differences between lengths of the blocks of $\mathsf{bw}(\lambda')$ and $\mathsf{bw}(\eta)$ corresponding to infinite clusters of t. Then it is easy to check that inequality (3.7) from Lemma 3.5.3 is fulfilled, because there is a summand on the left hand side of (3.7) which is equal to $\dim(\lambda', \eta)$. This summand corresponds to μ obtained from η by removing $-$ from a_j, appeared in the decomposition of $\mathsf{bw}(\eta)$ provided by Lemma 5.5.5. □

Proof of Theorem 5.7.5. Let us prove the first two parts of the theorem. We will do it by a single argument.

Lemma 5.7.6 implies that any strictly positive harmonic function on $\mathbb{Z}(t)$ is not finite. Then Theorem 3.3.14 provides a bijection between strictly positive indecomposable semifinite harmonic functions on $\mathbb{Z}(t)$ and strictly positive indecomposable finite and semifinite harmonic functions on $\mathbb{Z}(t)\backslash J(t)$. Recall that this bijection is defined by restriction of a function from $\mathbb{Z}(t)$ to $\mathbb{Z}(t)\backslash J(t)$. By Lemma 5.5.5 there is an injective homomorphism of graded graphs

$$\mathbb{Z}(t)\backslash J(t) \hookrightarrow \mathbb{Z}(t_1) \times \ldots \times \mathbb{Z}(t_k),$$

which image is an ideal. So, applying Theorem 3.3.14 once again we obtain a bijection between strictly positive indecomposable finite and semifinite harmonic functions on $\mathbb{Z}(t)\backslash J(t)$ and

$$\mathbb{Z}(t_1)\times\ldots\times\mathbb{Z}(t_k).$$

Propositions 4.2.6 and 5.4.4 yield that this graph admits a finite strictly positive indecomposable harmonic function. Then by Observation 5.3.3 and Proposition 3.4.6 it does not possess any strictly positive semifinite indecomposable harmonic functions. Hence the strictly positive indecomposable finite harmonic functions on this graph are in bijection with strictly positive finite and semifinite indecomposable harmonic functions on $\mathbb{Z}(t)\backslash J(t)$. In particular, this means that the graph $\mathbb{Z}(t)\backslash J(t)$ does not possess strictly positive semifinite indecomposable harmonic functions, because the aforementioned bijection is defined by the restriction of functions from the whole graph to an ideal which is isomorphic to $\mathbb{Z}(t)\backslash J(t)$.

Next, by Proposition 4.2.6, Theorem 5.4.3, and Proposition 5.4.4 each strictly positive indecomposable finite harmonic function on the graph

$$\mathbb{Z}(t_1)\times\ldots\times\mathbb{Z}(t_k)$$

is of the form

$$(\lambda(1),\ldots,\lambda(k))\mapsto F_{\lambda(1)}(v_1)\ldots F_{\lambda(k)}(v_k),$$

where v_i is a tuple of oriented consecutive subintervals of $(0,1)$ which orientations are defined by the signs of infinite clusters of t_i; the total length of all intervals of all v_i's equals 1.

Then each strictly positive finite indecomposble harmonic function on $\mathbb{Z}(t)\backslash J(t)$ is of the form

$$\lambda\mapsto F_{\lambda^{(1)}}(v_1)\ldots F_{\lambda^{(k)}}(v_k),$$

where $\lambda\mapsto(\lambda^{(1)},\ldots,\lambda^{(k)})$ is the map provided by Lemma 5.5.5.

The only thing we are left to do is to indicate how each of these functions should be extended from $\mathbb{Z}(t)\backslash J(t)$ to the whole graph $\mathbb{Z}(t)$. For that purpose we use Lemma 5.7.6.

To see that the functions $\varphi_{t,w}$ are distinct for distinct (t,w) we note that the functions

$$(\lambda(1),\ldots,\lambda(k))\mapsto F_{\lambda(1)}(v_1)\ldots F_{\lambda(k)}(v_k)$$

are distinct as functions on

$$\mathbb{Z}(t_1)\times\ldots\times\mathbb{Z}(t_k)$$

and by Theorem 3.3.14 the restriction of a function from this graph to an ideal isomorphic to $\mathbb{Z}(t)\backslash J(t)$ is a bijection between finite strictly positive indecomposable harmonic functions. □

5.8 Semifinite analog of the Vershik-Kerov ring theorem

Recall that by a semifinite analog of the ring theorem we mean Theorem 3.4.4. The following proposition describes the finite indecomposable harmonic function that is related to $\varphi_{t,w}$ by this theorem.

Theorem 5.8.1. *Let (t,w) be a semifinite zigzag growth model. For any $\mu\in\mathbb{Z}(t)\backslash J(t)$ and $\lambda\in\mathbb{Z}$ we have*

$$\varphi_{t,w}\left(F_\lambda F_\mu\right)=\varphi_w(F_\lambda)\varphi_{t,w}(F_\mu),$$

where φ_w is the finite harmonic function associated to the finitary oriented paintbox w, see Remark 5.7.2.

Before we start proving this proposition, we need to discuss one preparatory statement, which might be interesting in itself.

Recall the notation. The zigzag flange of t is denoted by $\mathsf{fl}(t) = (a_0, \ldots, a_k)$ and the splitting of t into sections looks like

$$t = (a_0,\ t_1,\ a_1,\ \ldots\ , a_{k-1},\ t_k,\ a_k).$$

The induced splitting of w reads as $w = v_1 \sqcup \ldots \sqcup v_k$.

Let us denote by $v_i(t_i)$ the collection of adjacent intervals on the real line corresponding to the finite template t_i and the tuple of numbers v_i. Namely, the length of the j-th interval in $v_i(t_i)$ equals $(v_i)_j$ and the orientation of this interval coincides with the sign of the j-th infinite cluster of t_i. It will be of no importance to us where the leftmost boundary point of $v_i(t_i)$ is placed; only lengths of intervals and their orientations matter.

Let ε be a real positive number. For any binary word a we denote by $\varepsilon(a)$ a tuple of adjacent oriented intervals each of which corresponds to a block of a; orientation of the interval is equal to sign of the block; all intervals are of length ε.

Then we define w_ε as a collection of adjacent intervals from $\varepsilon(a_0), \ldots, \varepsilon(a_k)$ and $v_1(t_1), \ldots, v_k(t_k)$ taken in the order proposed by the splitting of t into sections, that is

$$w_\varepsilon = \Big(\varepsilon(a_0), v_1(t_1), \varepsilon(a_1), v_2(t_2), \ldots, v_k(t_k), \varepsilon(a_k)\Big),$$

see Figure 5.8.

We do not specify where to place the leftmost boundary point of w_ε because it is not important in what follows. Note that the total length of w_ε equals $1 + O(\varepsilon)$.

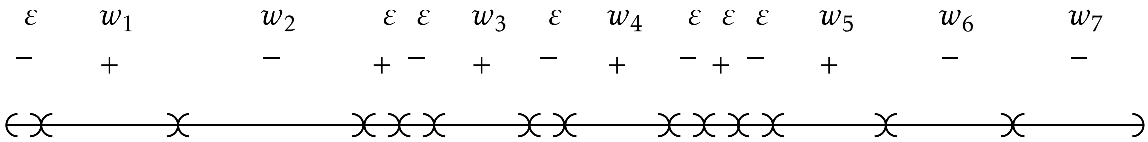


Figure. 5.8: The collection of adjacent intervals w_ε for the semifinite zigzag growth model (t, w) with the semifinite template

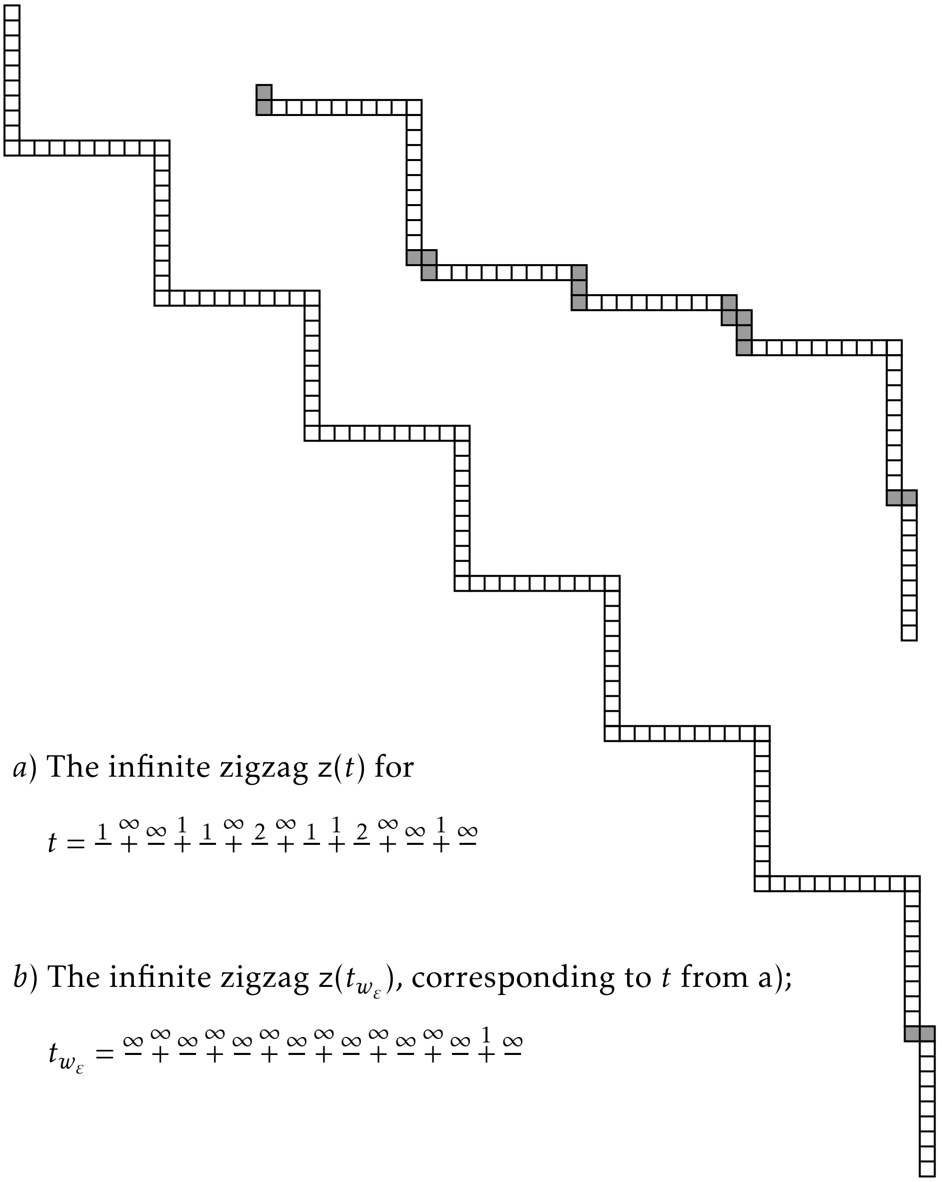
$$t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{1}{-}\overset{\infty}{+}\overset{2}{-}\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{2}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$$

from Figure 5.3. The length and orientation of an interval are indicated above the interval.

Furthermore, we can define a template t_{w_ε} in the same way as for oriented paintboxes, see the paragraph above Proposition 5.4.4. Then t_{w_ε} is a finite template and we may view w_ε as an infinite zigzag $\mathsf{z}(t_{w_\varepsilon})$ endowed with a tuple of real positive numbers, see Remark 5.4.6. Some of these real numbers equal ε while others come from w. The infinite rows and columns to which the number ε is assigned correspond to the zigzag flange of t. So, $\mathsf{z}(t_{w_\varepsilon})$ is obtained from $\mathsf{z}(t_w)$ by enlarging the rows and columns corresponding to the binary words from the zigzag flange. Namely, we replace these finite rows and columns with infinite ones, see Figure 5.9.

Let us denote by ν_t the zigzag obtained from $\mathsf{z}(t)$ by replacing each infinite row or column with a row or column of length 2. Then $\mathsf{bw}(\nu_t)$ is the binary word obtained from t by replacing each infinite symbol with a single symbol of the same sign.

Obviously, $\nu_t \in \mathbb{Z}(t) \setminus J(t)$ and $\mathbb{Z}(t)^{\nu_t}$, $\Big(\mathbb{Z}(t_{w_\varepsilon})\Big)^{\nu_t}$ are isomorphic to Pascal graphs of an appropriate dimensions. Recall that the superscript ν_t means that we take all zigzags greater than or equal to ν_t.



a) The infinite zigzag $\mathrm{z}(t)$ for

$$t = \overset{1}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{1}{-}\overset{\infty}{+}\overset{2}{-}\overset{\infty}{+}\overset{1}{-}\overset{1}{+}\overset{2}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$$

b) The infinite zigzag $\mathrm{z}(t_{w_\varepsilon})$, corresponding to t from a);

$$t_{w_\varepsilon} = \overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{\infty}{+}\overset{\infty}{-}\overset{1}{+}\overset{\infty}{-}$$

Figure. 5.9

Proposition 5.8.2. *There exists a natural number n, which depends only on t, such that for any* $\mu \in \left(\mathcal{Z}(t_{w_\varepsilon})\right)^{\nu_t}$ *we have*

$$\varphi_{t,w}(\mu) = \mathrm{const} \cdot \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} F_\mu(w_\varepsilon), \tag{5.7}$$

where const *does not depend on* μ, *but may depend on* (t, w); *recall that* $F_\lambda(w_\varepsilon)$ *is defined by Kerov's construction* (5.2), *see Section 5.4.2.*

Proof. To obtain the claim we should apply Lemma 5.4.7 to each multiple in Definition 5.7.3 and to the right hand side of (5.7). □

Remark 5.8.3. Equality (5.7) obviously holds for any $\mu \notin \mathcal{Z}(t_{w_\varepsilon})$, because in that case it turns into the trivial identity $0 = 0$.

Proof of Theorem 5.8.1. Let us denote by $c^\nu_{\lambda,\mu}$ the structure constants of multiplication in *QSym* written in the basis of fundamental quasisymmetric functions, i.e.

$$F_\lambda F_\mu = \sum_\nu c^\nu_{\lambda,\mu} F_\nu.$$

Note that $c^{\nu}_{\lambda,\mu} \geq 0$ and $c^{\nu}_{\lambda,\mu} \neq 0$ only if $\nu > \lambda, \mu$, see [17, p.35, (3.13)].

Then for any $\lambda \in \mathbb{Z}$ and $\mu \in \mathbb{Z}(t)^{\nu_t} \subset \left(\mathbb{Z}(t_{w_\varepsilon})\right)^{\nu_t}$ we can write

$$\varphi_{t,w}\left(F_\lambda F_\mu\right) = \sum_{\nu} c^{\nu}_{\lambda,\mu}\varphi_{t,w}(F_\nu) = \sum_{\nu} c^{\nu}_{\lambda,\mu} \operatorname{const} \cdot \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} F_\nu(w_\varepsilon) =$$
$$\operatorname{const} \cdot \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n}\left[\sum_{\nu} c^{\nu}_{\lambda,\mu} F_\nu(w_\varepsilon)\right] = \operatorname{const} \cdot \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon^n} F_\lambda(w_\varepsilon) F_\mu(w_\varepsilon)\right] =$$
$$\operatorname{const} \cdot \left[\lim_{\varepsilon\to 0} F_\lambda(w_\varepsilon)\right] \cdot \left[\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} F_\mu(w_\varepsilon)\right] = F_\lambda(w)\varphi_{t,w}\left(F_\mu\right).$$

We used formula (5.7) for $\varphi_{t,w}(F_\nu)$, because $\nu > \mu$ and then either $\nu \in \left(\mathbb{Z}(t_{w_\varepsilon})\right)^{\nu_t}$ or $\nu \notin \mathbb{Z}(t_{w_\varepsilon})$.

Thus, the finite indecomposable harmonic function ψ from Theorem 3.4.4 applied to $\varphi_{t,w}$ equals φ_w. $\square$

Chapter 6

Conclusion

6.1 Conclusion en français

En conclusion, cette thèse a abordé le problème de la classification des fonctions harmoniques semi-finies sur plusieurs graphes de branchement. Inspirée par la suggestion d'A.J. Wassermann d'utiliser une bijection entre les représentations de facteur fidèles d'une C^*-algèbre primitive A et celles d'un idéal fermé bilatère arbitraire de A pour la classification des traces semi-finies sur le groupe symétrique infini, une version combinatoire de la méthode de Wassermann a été développée. De plus, cette approche combinatoire a été appliquée pour décrire les fonctions harmoniques semi-finies sur le produit direct de graphes de branchement en termes de fonctions similaires sur les facteurs. La thèse se termine par l'exploration de la classification des fonctions harmoniques semi-finies sur le graphe en zigzag.

En résumé, cette thèse contribue à la compréhension et à la classification des fonctions harmoniques semi-finies sur les graphes de branchement. Le cadre combinatoire développé et l'application de la méthode de Wassermann fournissent des outils précieux pour aborder les problèmes de classification des traces sur les C^*-algèbres AF.

6.2 Conclusion in English

In conclusion, this thesis has addressed the problem of classifying semifinite harmonic functions on several branching graphs. Inspired by A.J. Wassermann's suggestion to use a bijection between faithful factor representations of a primitive C^*-algebra A and those of an arbitrary closed two-sided ideal of A for the classification of semifinite traces on the infinite symmetric group, a combinatorial version of Wassermann's method has been developed. Furthermore, this combinatorial approach has been applied to describe semifinite harmonic functions on the direct product of branching graphs in terms of similar functions on the factors. The thesis concludes by exploring the classification of semifinite harmonic functions on the zigzag graph.

In summary, this thesis contributes to the understanding and classification of semifinite harmonic functions on branching graphs. The developed combinatorial framework and the application of Wassermann's method provide valuable tools for addressing classification problems of traces on AF-algebras.

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Fonctions harmoniques sur les graphes ramifiés

La théorie classique des caractères des groupes finis et des groupes compacts peut être généralisée à d'autres classes de groupes et d'algèbres de diverses manières. Pour les groupes et les C^*-algèbres qui ne sont pas de type I, la théorie des caractères n'est pas liée aux représentations irréductibles mais aux représentations des facteurs normaux, c'est-à-dire aux homomorphismes des algèbres de von Neumann avec une trace finie ou semi-finie. Pour les AF-algèbres, on peut reformuler la théorie des caractères dans un langage combinatoire-algébrique grâce aux fonctions harmoniques non négatives sur les diagrammes de Bratteli. Les fonctions harmoniques qui prennent uniquement des valeurs finies sont en bijection avec les traces finies. Les traces semi-finies conduisent à des *fonctions harmoniques semi-finies*, qui peuvent prendre la valeur $+\infty$ de telle manière que ces valeurs infinies puissent être approximées par des valeurs finies. Pour classer les traces semi-finies sur le groupe symétrique infini, A.J.Wassermann a suggéré en 1981 d'utiliser une bijection entre les représentations de facteurs fidèles d'une C^*-algèbre primitive A et celles d'un idéal bilatère fermé arbitraire de A. Nous développons une version combinatoire de la méthode de Wassermann. Ensuite, nous l'appliquons pour décrire les fonctions harmoniques semi-finies sur le produit direct de graphes de branchement en termes de fonctions similaires sur les facteurs. La dernière partie de la thèse est consacrée à la classification des fonctions harmoniques semi-finies sur le graphe en zigzag.

Mots clés: Diagrammes de Bratteli, graphes ramifiés, fonctions harmoniques, traces semi-finies, algèbres AF, mesures invariantes.

Harmonic functions on branching graphs

Classical character theory of finite and compact groups may be generalized to other classes of groups and algebras in various ways. For groups and C^*-algebras *not* of type I the character theory is related *not* to irreducible representations but to normal factor representations, i.e. homomorphisms to von Neumann algebras with a finite or semifinite trace. For AF-algebras one can reformulate the character theory in a combinatorial-algebraic language, speaking about non-negative harmonic functions on Bratteli diagrams. Harmonic functions that take only finite values are in bijection with finite traces. Semifinite lower semicontinuous traces lead to *semifinite harmonic functions*, which may take the value $+\infty$ in such a way that these infinite values can be approximated by finite ones. In order to classify the semifinite traces on the infinite symmetric group A.J.Wassermann suggested in 1981 to use a bijection between faithfull factor representations of a primitive C^*-algebra A and those of an arbitrary closed two-sided ideal of A. We develop a combinatorial version of Wassermann's method. Subsequently, we apply it to describe semifinite harmonic functions on the direct product of branching graphs in terms of similar functions on the factors. The last part of the thesis is devoted to classification of semifinite harnomic functions on the zigzag graph.

Keywords: Bratteli diagrams, branching graphs, harmonic functions, semifinite traces, AF-algebras, invariant measures.

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